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ELEMENTARY TRIGONOMETRY.

ELEMENTARY TRIGONOMETRY.

BY

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AND

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ASSISTANT MASTERS AT ETON COLLEGE.

CAMBRIDGE:
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1891.

PREFACE.

This work is intended primarily for use as a school text-book. We hope it will meet the wants of all who are beginning Trigonometry, and be found adequate for the majority of Army Candidates.

Our index of contents will show that we have departed somewhat from the recognised treatment of the subject, but we believe that our arrangement will prove suitable for most beginners, and minimise their difficulties. Those who disapprove of it will find no difficulty in directing the chapters to be read in an order more in accordance with their own views or the capacities of individual pupils.

J. M. D.
R. H. W.

Eton College,
October, 1891.

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PART I.

CHAPTER I.

RECTANGULAR MEASURE.

1. Definition.—Trigonometry is the Science which has for its principal aim the **Measurement of the Sides and Angles of Triangles.**

2. The **Sides of Triangles** are measured by **Numbers** expressing their **Lengths.**

These **Numbers** are often incommensurable quantities, such as surds.

3. In Trigonometry **an angle** P O Q is said to measure the inclination of two straight lines **O P, O Q,** by **the amount which O P has revolved round O starting from an initial position O Q.** Thus, as will be shown more fully later, an angle may be of any size, as O P may have made any number of complete revolutions round O before taking up its final position, each complete revolution being through 4 right angles.

Our proofs, to the end of page 90, will **only apply to angles less than a right angle.**

4. Two systems are used for measuring **Angles.**

1. **Rectangular Measure,** in which every angle is expressed as a multiple or a part of a **Right Angle.**
2. **Circular Measure,** which will be defined in Chapter II.

5. The **Right Angle** may be subdivided

(1) by the **Sexagesimal**, sometimes called the English method,

(2) by the **Centesimal**, sometimes called the French method.

In the *Sexagesimal* system

1 Right angle	= 90 Degrees (90°).
1 Degree	= 60 Minutes ($60'$).
1 Minute	= 60 Seconds ($60''$).

In the *Centesimal* system

1 Right angle	= 100 Grades (100^g).
1 Grade	= 100 French Minutes ($100^{\backprime}$).
1 French Minute	= 100 French Seconds ($100^{\backprime\backprime}$).

Thus an angle of 17 degrees 43 minutes 25 seconds is written $17^\circ\,43'\,25''$. An angle of 5 grades 77 French minutes 81 French seconds is written $5^g\,77^{\backprime}\,81^{\backprime\backprime}$.

In the centesimal system the number of grades, &c., in an angle can be directly expressed as the **decimal of a right angle.**

Thus $\quad 5^g\,77^{\backprime}\,81^{\backprime\backprime} = 5^g\,77\cdot81^{\backprime} = 5\cdot7781^g$

$= \cdot057781$ of a right angle.

Note.—For practical purposes the sexagesimal method of measuring angles is now universally employed.

EXAMPLES I (*a*).

Express the following fractions of a right angle in degrees, minutes, and seconds; also in grades, Fr. minutes, and Fr. seconds—

(1) $\frac{2}{3}$. (2) $\frac{5}{27}$. (3) $\frac{11}{16}$. (4) $\frac{1}{625}$.
(5) $\cdot375$. (6) $\cdot93125$. (7) $1\cdot1458\dot{3}$. (8) $2\cdot38725$.

Express as decimals of a right angle—

(9)	$13^\circ\,30'$.	(15)	$12^g\,17^{\backprime}\,24^{\backprime\backprime}$.
(10)	$45^\circ\,33'\,45''$.	(16)	$2^g\,18^{\backprime}\,24^{\backprime\backprime}$.
(11)	$5^\circ\,45'\,36''$.	(17)	$73^g\,7^{\backprime}\,40^{\backprime\backprime}$.
(12)	$17^\circ\,6'\,27''$.	(18)	$3^g\,30^{\backprime}\,4^{\backprime\backprime}$.
(13)	$89136''$.	(19)	$327^{\backprime}$.
(14)	$3915'$.	(20)	$17^{\backprime\backprime}$.

(21) Express in grades, &c., the following decimal fractions of a Right angle—$\cdot3452\dot{7}$, $\cdot021004$, $\cdot2\dot{0}\dot{7}$, $\cdot2\dot{5}3\dot{8}$.

6. Let D be the number of **degrees**, G the number of **grades** in any angle.

$$\text{Now D degrees} = \frac{D}{90} \text{ of a right angle,}$$

$$\text{and} \quad \text{G grades} = \frac{G}{100} \quad \text{''} \quad \text{''}$$

$\therefore$ $\frac{D}{90}$ and $\frac{G}{100}$ represent the fraction which the original angle is of a right angle and are therefore **equal.**

$$\therefore \quad \frac{D}{90} = \frac{G}{100},$$

whence $$D = \frac{9}{10}G, \text{ and } G = \frac{10}{9}D.$$

7. To change from one system of measurement into the other; make the given angle into a decimal of a right angle and then reduce to the other system.

This will be best understood from the following examples:—

Example (1).—To express 31° 43′ 30″ in grades, &c. Reduce 31° 43′ 30″ to the decimal of a right angle.

$$\begin{array}{r|l} 60 & 30 \text{ seconds} \\ 60 & 43 \cdot 5 \text{ minutes} \\ 90 & 31 \cdot 725 \text{ degrees} \\ & \cdot 3525 \text{ right angles.} \end{array}$$

$$\begin{aligned} \therefore \quad 31^\circ\, 43'\, 30'' &= \cdot 3525 \text{ of a right angle;} \\ &= 35 \cdot 25^g \\ &= 35^g\ 25^{\backprime}. \end{aligned}$$

Example (2).—To express 73° 41′ 14″ in the French system.

$$\begin{array}{r|l} 60 & 14 \\ 60 & 41 \cdot 2\dot{3} \\ 90 & 73 \cdot 687\dot{2} \\ & \cdot 818\dot{7}469135802\dot{}; \end{array}$$

$$\begin{aligned} \therefore \quad 73^\circ\, 41'\, 14'' &= \cdot 818\dot{7}46913580\dot{2} \text{ of a right angle} \\ &= 81^g\ 87^{\backprime}\ 46 \cdot \dot{9}1358024\dot{6}^{\backprime\backprime}. \end{aligned}$$

Example (3).—To express $81^{g}\ 72^{\backprime}\ 5^{\backprime\backprime}$ in the English system.

$$81^{g}\ 72^{\backprime}\ 5^{\backprime\backprime} = \cdot 817205 \text{ of a right angle.}$$

$$\begin{array}{r}
\cdot 817205 \\
90 \\
\hline
73 | \cdot 548450 \text{ degrees} \\
60 \\
\hline
32 | \cdot 90700 \text{ minutes} \\
60 \\
\hline
54 \cdot 42 \text{ seconds}
\end{array}$$

$$\therefore \quad 81^{g}\ 72^{\backprime}\ 5^{\backprime\backprime} = 73^{\circ}\ 32'\ 54 \cdot 42''.$$

Example (4).—Express $7^{g}\ 2^{\backprime}\ 50^{\backprime\backprime}$ in the English system.

$$7^{g}\ 2^{\backprime}\ 50^{\backprime\backprime} = \cdot 07025 \text{ of a right angle.}$$

$$\begin{array}{r}
\cdot 07025 \text{ of a right angle} \\
90 \\
\hline
6 | \cdot 32250 \text{ degrees} \\
60 \\
\hline
19 | \cdot 3500 \text{ minutes} \\
60 \\
\hline
21 \cdot 00 \text{ seconds.}
\end{array}$$

$$\therefore \quad 7^{g}\ 2^{\backprime}\ 50^{\backprime\backprime} = 6^{\circ}\ 19'\ 21''.$$

EXAMPLE I (*b*).

Transform to the French system—

(1) $29^{\circ}\ 5'\ 33''$.

(2) $78^{\circ}\ 36'\ 54''$.

(3) $10^{\circ}\ 51'\ 22 \cdot 5''$.

(4) $66^{\circ}\ 27'\ 42''$.

(5) $33^{\circ}\ 18'\ 54''$.

(6) $27^{\circ}\ 30'\ 45''$.

Transform to the English system—

(7) $17^{g}\ 62^{\backprime}\ 50^{\backprime\backprime}$.

(8) $10^{g}\ 1^{\backprime}\ 25^{\backprime\backprime}$.

(9) $54^{g}\ 10^{\backprime}\ 44^{\backprime\backprime}$.

(10) $31^{g}\ 15^{\backprime}\ 65^{\backprime\backprime}$.

(11) $42^{g}\ 3^{\backprime}\ 5^{\backprime\backprime}$.

(12) $7^{g}\ 20^{\backprime}\ 8^{\backprime\backprime}$.

(13) Express the angles of a regular hexagon in degrees and grades.

(14) In a right-angled triangle one of the acute angles is $36^g\ 87'\ 50''$. Find the other in degrees, &c.

(15) In an isosceles triangle the vertical angle measures 15°. Express the base angles in grades, &c.

(16) Find the size in degrees and grades of the angles of an isosceles triangle, in which each of the angles at the base is double of the remaining angle.

(17) Also when each of the equal angles is $\frac{1}{4}$ of the remaining angle.

(18) A, B, C are the angles of a triangle; if $A = \dfrac{B}{2} = \dfrac{C}{3}$, express them in degrees and grades.

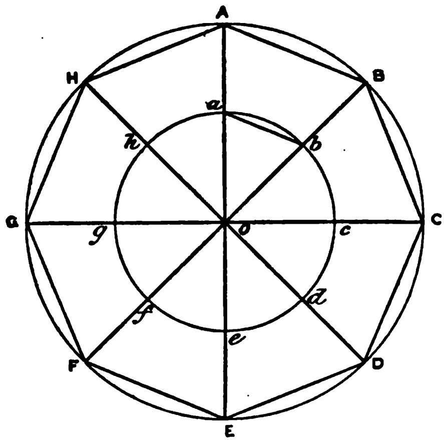
(19) One angle of a triangle measures 37° 48′ and another $103^g\ 73'\ 60''$. Express the third angle in degrees and grades.

(20) The angles of a triangle are in arithmetical progression. The number of grades in the greatest : number of degrees in the sum of the other two : : 10 : 11. Find the angles in degrees and grades.

CHAPTER II.

CIRCULAR MEASURE.

8. Preliminary Theorem. I. *The circumferences of circles vary as their radii.*



Let ABCDEFGH be a regular polygon of n sides. Let O be the centre of the circumscribing circle. Join A O, B O, C O, &c. With centre O and any radius describe another circle; the points $abcdefgh$ where this circle meets the radii A O, B O, C O, &c., will form another polygon similar to the first one.

Let R be the radius of the larger, r of the smaller circle. Then in the similar triangles A B O, $a\,b$ O we have

$$\frac{AB}{AO} = \frac{ab}{aO}, \text{ or } \frac{AB}{R} = \frac{ab}{r}.$$

Therefore
$$\frac{n \cdot AB}{R} = \frac{n \cdot ab}{r};$$

or
$$\frac{\text{perimeter of } ABCDEFGH}{R} = \frac{\text{perimeter of } abcdefgh}{r}.$$

This is true, whatever be the value of n. But if we increase n, then the greater it becomes, the smaller do the sides of the polygon become, and in the limit the perimeter of the polygon

becomes approximately the circumference of the circle in which it is inscribed. Therefore in the limiting case we may say

$$\frac{\text{circumference ABCDEFGH}}{R} = \frac{\text{circumference } abcdefgh}{r}.$$

And as this fraction is thus proved the same for any two circles it must be the same for all circles.

That is $\dfrac{\text{circumference of a circle}}{\text{radius}}$ is a constant.

This constant is called 2π; or in other words,

$$\text{circumference} = 2\pi r,$$
$$\text{semi-circumference} = \pi r,$$
$$\text{quadrant} = \frac{\pi r}{2}.$$

π is found to be a non-terminating, non-recurring decimal. Calculated to 5 places, it is $3 \cdot 14159$. A less accurate approximation to its value is $\frac{22}{7}$.

Example.—A wheel has made 840 revolutions in a mile; find its diameter, π being assumed to be $\frac{22}{7}$.

Since the wheel revolves 840 times in a mile, its circumference in feet must be $\frac{5280}{840}$.

$\therefore$ Its diameter must be $\dfrac{5280}{840} \div \pi = \dfrac{5280}{840} \times \dfrac{7}{22} = 2$ ft.

EXAMPLES II (*a*).

$$\left(\text{Take } \pi \doteq \frac{22}{7}\right).$$

(1) The driving wheel of an engine going 60 miles an hour makes 4 revolutions per second. Find the diameter of the wheel.

(2) How many revolutions per mile will a wheel make whose radius is 3 ft?

(3) How many miles per hour is a cart going when its wheels, 5 ft. in diameter, make 28 revolutions per minute.

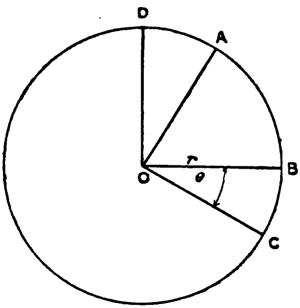
(4) Assuming the earth to describe round the sun, in 365 days, a circle of which the radius is 91,250,000 miles, find how far the earth travels in 1 hour.

(5) Find the difference in length between the circumference of a circle of radius 7 ft., and the perimeter of a regular hexagon inscribed in it.

(6) Supposing the earth to be a sphere 7980 miles in diameter; find the distance between two consecutive parallels of latitude.

9. Theorem II. *The angle at the centre of a circle subtended by an arc equal to the radius is constant.*

Let the angle A O B be such an angle, so that the arc A B = O B = r.



And let D O B be a right angle.

Now by Theorem I. above

$$DB = \frac{\pi r}{2}.$$

And by Euclid vi. 33, angles at the centre of a circle are proportional to the arcs on which they stand.

$$\therefore \frac{\angle AOB}{\angle DOB} = \frac{AB}{DB} = \frac{r}{\frac{\pi r}{2}} = \frac{2}{\pi};$$

or $\angle AOB = \frac{2}{\pi}$ times $\angle DOB = \frac{2 \text{ right angles}}{\pi}$ which is constant. Using the approximate value $\pi = 3 \cdot 14159$, it is found to be $57 \cdot 2957^\circ$.

10. Definition. This angle at the centre of a circle, subtended by an arc equal in length to the radius, is called a **Radian.**

11. The Radian is taken as a standard for a third system of measuring angles, called **circular measure.** One Radian is written 1^c.

It is not arbitrarily sub-divided as the units in the other systems, but all angles are directly expressed as parts or multiples of it.

Let C O B be any angle θ, A O B a radian as above,

Then $$\frac{\theta}{\angle AOB} = \frac{\text{arc CB}}{\text{arc AB}} = \frac{\text{arc}}{\text{radius}} \quad \therefore \theta = \frac{\text{arc}}{\text{radius}} \times \angle AOB.$$

That is $$\theta = \frac{\text{arc}}{\text{radius}} \text{ times 1 radian,}$$

so that the fraction $\frac{\text{arc}}{\text{radius}}$ expresses the measure of the angle in radians.

Example 1.—An angle at the centre of a circle whose radius is 9 ft. stands on an arc of 6 ft.

Its circular measure is $\frac{\text{subtending arc}}{\text{radius}} = \frac{6^c}{9} = \frac{2^c}{3}$.

Example 2.—The circular measure of an angle is $\frac{3}{7}$. It is at the centre of a circle whose radius is 4 ft. Find the length of the subtending arc.

Let x be the length of the arc required.

Since circular measure of angle $= \frac{\text{subtending arc}}{\text{radius}}$,

we have $$\frac{3}{7} = \frac{x}{4} \text{ or } x = \frac{12}{7} \doteq 1\tfrac{5}{7} \text{ ft.}$$

EXAMPLES II (*b*).

$$\left(\text{Take } \pi = \frac{22}{7}.\right)$$

(1) Find the radius of a circle when an arc of it 3 ft. long subtends an angle of $\cdot25^c$ at the centre.

(2) Find the length of an arc subtending an angle whose circular-measure is $\frac{7}{30}$, at the centre of a circle of 9 inches radius.

(3) What is the circular measure of an angle subtended at the centre of a circle of radius 15 inches by an arc of length 2 ft.?

(4) The circumference of a circle is 2200 miles. What is the circular measure of the angle at its centre subtended by an arc 420 miles long?

(5) Find the circumference of a circle in which an arc $7\frac{1}{2}$ ft. long subtends at the centre an angle whose circular measure is $\cdot 375$.

(6) A railway train is travelling along a curve of $\frac{1}{3}$ mile radius at the rate of 25 miles per hour; through what angle (in circular measure) will it turn in half a minute?

(7) On a curve of 1 mile radius a train is observed to turn through an angle of $\cdot 075^c$ in 15 seconds; what is its rate?

(8) The circumference of a circle measures 484 ft.; what is the length of an arc of it which subtends at the centre an angle of $\cdot 375$ radians?

12. It is easy to convert angles in circular measure to the other two systems, or *vice versâ*. For by § 9 the unit of circular measure (1 radian) $= \dfrac{2 \text{ right angles}}{\pi}$.

That is, π radians $= 2$ right angles $= 180°$ or 200^g.

These conditions enable us to convert as in the following examples:—

(1) What is the circular measure of an angle of 36°?

$$180° = \pi^c$$

$$1° = \frac{\pi^c}{180}$$

$$36° = \frac{36\,\pi^c}{180} = \frac{\pi^c}{5}.$$

That is, the circular measure of an angle of 36° is $\frac{\pi}{5}$ or $\frac{22}{35}$.

(2) What is the circular measure of an angle of $9° \, 13' \, 30''$?

Express the angle as a decimal of a degree. It becomes $9 \cdot 225°$.

Then as before $$180° = \pi^c$$

$$1° = \frac{\pi^c}{180}$$

$$9 \cdot 225° = \frac{9 \cdot 225 \, \pi^c}{180} = \frac{9225 \, \pi^c}{180000} = \frac{41 \, \pi^c}{800}.$$

N.B.—The expressions for angles in circular measure are often left in this form without substituting the value of π; and in this case the symbol ^{c} is generally omitted.

(3) Express in the French system the angle whose circular measure is $\frac{5}{8}$.

We have $\pi^c = 200^g$

$$1^c = \frac{200^g}{\pi}$$

$$\frac{5^c}{8} = \frac{200}{\pi} \times \frac{5^g}{8} = \frac{125 \times 7}{22} \text{ grades} = 39 \cdot 7\dot{7}\dot{2} \text{ grades}$$
$$= 39^g \, 77' \, 27'' \cdot \dot{2}\dot{7}.$$

13. Let D be the number of degrees,
G " " grades,
R " " radians, in any angle.

Then $\frac{D}{90}$, $\frac{G}{100}$, $\frac{R}{\frac{\pi}{2}}$, are each expressions for the fraction of a right angle which it represents;

$$\therefore \quad \frac{D}{90} = \frac{G}{100} = \frac{2\,R}{\pi}.$$

By means of these formulæ we can readily pass from one system of measurement to the other.

Example.—If $G = 33 \cdot \dot{3}$, $\frac{G}{100} = \frac{1}{3}$; that is, $\frac{D}{90} = \frac{2\,R}{\pi} = \frac{1}{3}$.

$$\therefore \quad D = 30°, \; R = \frac{\pi}{6}.$$

EXAMPLES II (*c*).

Find the circular measure of the following angles (in terms of π)—

(1) $24°$.

(2) $63°$.

(3) $4° 30'$.

(4) $6° 12' 36''$.

(5) $78° 7' 30''$.

(6) 24^g.

(7) $51^g 25'$.

(8) $15^g 62' 50''$.

(9) $23^g 43' 75''$.

(10) $80^g 6' 25''$.

Express the following angles in both rectangular systems—

(11) $\frac{\pi}{6}$. (12) $\frac{3\pi}{8}$. (13) $\frac{11}{28}^c$. (14) $1\frac{4}{7}^c$.

(15) $\frac{17\pi}{64}$. (16) $\cdot 4^c$. (17) $1\frac{1}{4}^c$. (18) $\frac{4\pi}{75}$.

(19) $1\cdot 65^c$. (20) θ^c.

(21) The sum of two angles in circular measure is $\frac{\pi}{2}$, and their difference in degrees is $30°$: find them.

(22) The circular measure of the difference between the vertical angle of an isosceles triangle and either of the base angles is $\frac{\pi}{4}$. Express the angles in degrees. (The vertical angle is the smallest.)

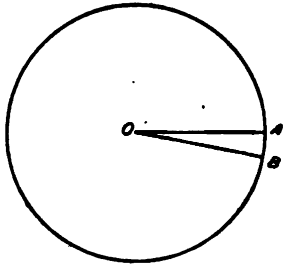
(23) Find an angle such that the number of degrees in it shall be 70 more than the number which is its circular measure.

(24) What is the circular measure of an angle subtended at the centre of a circle of 1 ft. diameter by an arc equal in length to the radius of a circle 1 ft. in circumference?

(25) The angles of a plane triangle are x degrees, x grades, and $\frac{\pi x}{90}$ radians respectively. Find x.

(26) Supposing the earth to move round the sun in 365 days uniformly in a circle, find the circular measure of the angle described in a day.

14. In many practical applications of the foregoing principles we may assume, by employing reasoning similar to that used in § 8, that a very small arc is equal in length to the chord which subtends it. That is, if the angle A O B is very small, we assume that its circular measure is $\frac{\text{chord A B}}{\text{O B}}$ instead of $\frac{\text{arc A B}}{\text{O B}}$.



Example.—How large a mark approximately on a target 1000 yds. off will subtend an angle of 1′ at the eye?

The circular measure of an angle of 1′ is $\frac{\pi}{180 \times 60}$.

Let x be the length of the mark. We assume

$$\frac{x}{1000} = \frac{\pi}{180 \times 60}.$$

$$\therefore\ x = \frac{10\pi}{18 \times 6} \text{ yds.} = \frac{5\pi}{18} \text{ ft.} = \frac{5 \times 22}{18 \times 7} \text{ ft.} = \frac{55}{63} \text{ ft.} = 10\tfrac{10}{21} \text{ in.}$$

EXAMPLES II (*d*).

(1) Assuming the earth's diameter to be 8000 miles, and its distance from the sun to be 92,000,000 miles, find to the nearest second the angle which it subtends at the sun.

(2) Find the distance at which a person looking towards the sun must hold a coin $\frac{1}{2}$ inch in diameter that it may just hide the sun; assuming the angle subtended by the sun's diameter at the eye to be 32′.

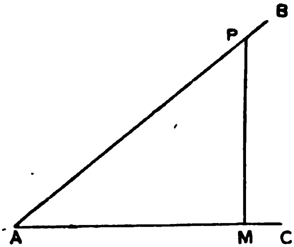
(3) What is approximately the diameter of a planet 54,000,000 miles distant which subtends an angle of $16 \cdot 8''$ at the eye?

(4) What is approximately the diameter of the sun if at a distance of 92,000,000 miles it subtends an angle of 32′?

CHAPTER III.

TRIGONOMETRICAL RATIOS OF ANGLES LESS THAN A RIGHT ANGLE.

15. Let B A C be any angle less than a right angle. In A B, one of the straight lines bounding the angle A, take any point P, and draw P M perpendicular to the other side A C.



If the angle P A M (called A) is under consideration, A P, the side subtending the right angle, is called the hypothenuse; P M, the side opposite the angle A, is called the perpendicular; A M, the side adjacent to the angle A, is called the base.

Then (1) $\dfrac{PM}{AP}$, *i.e.* $\dfrac{\text{perpendicular}}{\text{hypothenuse}}$ is called the **sine of A** (written **sin A**).

(2) $\dfrac{AM}{AP}$, *i.e.* $\dfrac{\text{base}}{\text{hypothenuse}}$ is called the **cosine of A** (**cos A**).

(3) $\dfrac{PM}{AM}$, *i.e.* $\dfrac{\text{perpendicular}}{\text{base}}$ is called the **tangent of A** (**tan A**).

(4) $\dfrac{AP}{PM}$, *i.e.* $\dfrac{\text{hypothenuse}}{\text{perpendicular}}$ is called the **cosecant of A** (**cosec A**).

(5) $\dfrac{AP}{AM}$, *i.e.* $\dfrac{\text{hypothenuse}}{\text{base}}$ is called the **secant of A** (**sec A**).

(6) $\dfrac{AM}{PM}$, *i.e.* $\dfrac{\text{base}}{\text{perpendicular}}$ is called the **cotangent of A** (**cot A**).

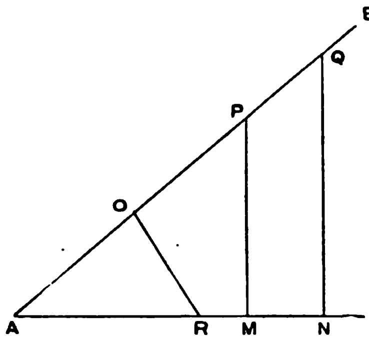
16. 1 − cos A is called the versed sine of A, written vers A; and 1 − sin A is called the coversed sine of A, written covers A; but these terms are seldom used.

17. It will be seen at once from these definitions that

$$\text{cosec } A = \frac{1}{\sin A}, \ \sec A = \frac{1}{\cos A}, \text{ and } \cot A = \frac{1}{\tan A}.$$

18. Each of the above ratios is a **number**, because each is the result of dividing one length by another.

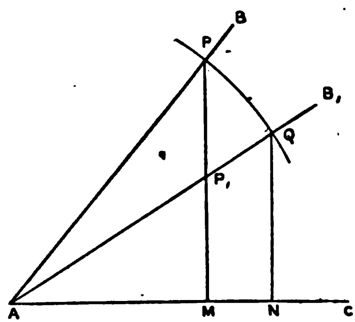
19. The ratios remain invariable for a given angle. For from any other point Q in A B draw Q N perpendicular to A C, and from any point R in A C draw R O perpendicular to A B. Then P A M, Q A N, and R A O are similar triangles.



$$\therefore \frac{QN}{AQ} = \frac{RO}{AR} = \frac{PM}{AP}.$$

That is, these three ratios, which by the definition are all called sin A, have the same value. So for the other trigonometrical ratios.

20. The ratios vary when the angle under consideration varies. For let B A C, B′ A C, be two angles, and from P draw P P′ M perpendicular to A C.



Then A P′ P is an obtuse angle, being greater than the angle A M P′ (Euclid I. xvi.).

∴ A P P′ is an acute angle.

∴ A P is greater than A P′ (Euclid I. xix.).

So we have $\frac{AM}{AP}$ less than $\frac{AM}{AP'}$; that is, cos B A C less than cos B′ A C.

Again, cutting off A Q on A B′ equal to A P, and drawing Q N perpendicular to A C, we have Q N less than P M,

and so $$\frac{PM}{AP} \text{ greater than } \frac{QN}{AQ};$$

or, sin B A C greater than sin B′ A C.

And $$\frac{PM}{AM} \text{ is greater than } \frac{P'M}{AM};$$

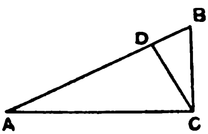
or, tan B A C is greater than tan B′ A C.

EXAMPLES III (*a*).

(1) Write down the values of the six trigonometrical ratios, when in the figure in § 15, A P = 10, P M = 8, A M = 6.

(2) Under the same conditions find the value of

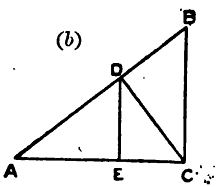
$$5 \sin A + 3 \tan A - 7 \sec A.$$



(3) From the adjoining figure (the angles A C B, A D C, B D C being right angles) write down the expressions for cos A C D, and tan B C D, and two expressions for sin A B C and cot B A C.

(4) In the same figure, if A B = 13, A C = 12, and B C = 5, find the value of tan A B C + sin B C D + cosec D A C, and of sec C A D − cos D B C + cot D C A.

(*b*)



(5) Write down all the possible expressions for all the trigonometrical ratios of the angle A in Fig. (*b*), there being right angles at C, D, and E.

(6) If A B C is a triangle with a right angle at C, and if A C = 12 and B C = 9, find the values of sin A, sin B, sec A, tan A, and versin A.

(7) If in a right-angled triangle the hypothenuse is 26 inches long and one of the sides is 10, write down all the trigonometrical ratios for both the acute angles.

(8) If the difference between the hypothenuse and one of the sides of a right-angled triangle be half the other side, prove that the sine of the smallest angle is $\frac{3}{5}$.

21. We have seen that the ratios sin A, cos A, tan A, &c., are numbers which vary in value as the angle varies; they can therefore be treated as algebraical quantities, a, b, c, d, e, f. For instance, $\sin A \times \sin A = (\sin A)^2$, just as $a \times a = a^2$; again $(\sin A + \cos A) \times (\sin A - \cos A) = (\sin A)^2 - (\cos A)^2$, just as $(a + b) \times (a - b) = a^2 - b^2$.

Note.—$(\text{Sin } A)^2$ is more shortly written $\sin^2 A$, &c.; thus $(\sin A)^2 + (\cos A)^3$ is written $\sin^2 A + \cos^3 A$.

22. The ratios sin A, cos A, tan A, &c., are not independent of one another.

We have seen in § 17 that

(1) $\text{Cosec } A = \dfrac{1}{\sin A}$,

(2) $\text{Sec } A = \dfrac{1}{\cos A}$,

(3) $\text{Cot } A = \dfrac{1}{\tan A}$.

We shall now prove four more relations between the ratios.

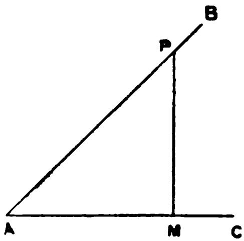
(4) $\text{Tan } A = \dfrac{\sin A}{\cos A}$,

(5) $\text{Cos}^2 A + \sin^2 A = 1$,

(6) $\text{Sec}^2 A = 1 + \tan^2 A$,

(7) $\text{Cosec}^2 A = 1 + \cot^2 A$.

N.B.—These two last are not independent, but can be deduced from the other five.



23. To prove $\tan A = \frac{\sin A}{\cos A}$,

we have

$$\tan A = \frac{PM}{AM} = \frac{PM}{AP} \times \frac{AP}{AM}$$

$$= \frac{PM}{AP} \div \frac{AM}{AP}$$

$$= \sin A \div \cos A = \frac{\sin A}{\cos A}.$$

24. To prove $\sin^2 A + \cos^2 A = 1$.

With figure as before, we have, since P A M is a right-angled triangle,

$$PM^2 + AM^2 = AP^2; \quad . \quad . \quad . \quad (a)$$

divide each term of this relation by AP^2 and we obtain

$$\frac{PM^2}{AP^2} + \frac{AM^2}{AP^2} = 1,$$

which may be written

$$\left(\frac{PM}{AP}\right)^2 + \left(\frac{AM}{AP}\right)^2 = 1;$$

or,

$$\sin^2 A + \cos^2 A = 1.$$

25. To prove $\sec^2 A = 1 + \tan^2 A$.

Divide each term in (*a*) above by AM^2.

We have then

$$\frac{AP^2}{AM^2} = 1 + \frac{PM^2}{AM^2},$$

that is,

$$\left(\frac{AP}{AM}\right)^2 = 1 + \left(\frac{PM}{AM}\right)^2;$$

or,

$$\sec^2 A = 1 + \tan^2 A.$$

26. To prove $\operatorname{cosec}^2 A = 1 + \cot^2 A$.

Divide each term in (*a*) by PM^2.

We have then

$$\frac{AP^2}{PM^2} = 1 + \frac{AM^2}{PM^2},$$

that is,

$$\left(\frac{AP}{PM}\right)^2 = 1 + \left(\frac{AM}{PM}\right)^2;$$

or,

$$\operatorname{cosec}^2 A = 1 + \cot^2 A.$$

Note.—In what precedes A stands for any angle whatever

less than a right angle, ; we may therefore substitute any other symbol for A; for example, it is true that

$$\sin^2 3\,B + \cos^2 3\,B = 1,$$

$$\sec^2 \frac{B}{2} = 1 + \tan^2 \frac{B}{2},$$

$$\operatorname{cosec}^2 (B + C) = 1 + \cot^2 (B + C).$$

27. The relations we have established enable us to express any one of the trigonometrical ratios in terms of any other.

Example 1.—Express sin A in terms of tan A.

$$\frac{\sin A}{\cos A} = \tan A \text{ (by § 23)};$$

$$\therefore \quad \sin A = \tan A \cos A = \frac{\tan A}{\sec A} \text{ (by § 17)},$$

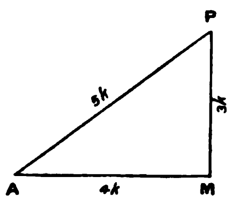
$$= \frac{\tan A}{\sqrt{1 + \tan^2 A}} \text{ (by § 25)}.$$

Example 2.—To express cosec A in terms of cos A.

$$\operatorname{cosec} A = \frac{1}{\sin A} \text{ (by § 17)}, \quad = \frac{1}{\sqrt{1 - \cos^2 A}} \text{ (by § 24)}.$$

Example 3.—Given $\sin A = \frac{3}{5}$ to deduce the other trigonometrical ratios.

It is given that $\frac{PM}{AP} = \frac{3}{5}$.



$$\therefore \quad \frac{PM}{3} = \frac{AP}{5} = k, \text{ suppose};$$

$$\therefore \quad PM = 3\,k, \quad AP = 5\,k.$$

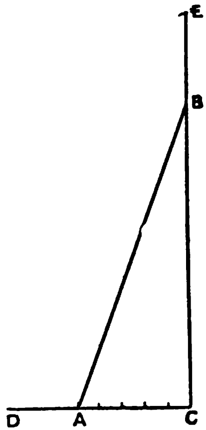
Now $\qquad AP^2 = AM^2 + PM^2;$

$$\therefore \quad 25\,k^2 = AM^2 + 9\,k^2,$$

whence $\qquad AM = 4\,k.$

$$\therefore \quad \cos A = \frac{4\,k}{5\,k} = \frac{4}{5}, \quad \tan A = \frac{3\,k}{4\,k} = \frac{3}{4}, \text{ \&c.}$$

Example 4.—To construct an angle whose cosine is $\frac{5}{13}$. Draw two lines C D, C E at right angles to one another. From C



mark off 5 equal parts along C D; let C A be the length. With centre A and radius equal to 13 of the parts describe a circle cutting C E in B.

B A C will be the required angle, since

$$\cos BAC = \frac{AC}{AB} = \frac{5}{13}.$$

EXAMPLES III (*b*).

(1) Express all the other trigonometrical ratios in terms of the tangent.

(2) Express the cotangent of an angle in terms of the cosine.

(3) Express all the other ratios in terms of the sine.

Find the other trigonometrical ratios in the following cases :—

(4) Tan A $= \frac{3}{4}$.

(5) Cos A $= \frac{99}{101}$.

(6) Sin A $= \frac{5}{13}$.

(7) Cot A $= \frac{56}{33}$.

(8) Sec A $= \frac{25}{7}$.

(9) If sin A $= \frac{4}{5}$, prove that tan A + sec A = 3.

(10) If cosec A $= \frac{17}{8}$, find the value of $\frac{4 \cot A + 5 \sec A}{3 \cos A + 8 \sin A}$.

Construct geometrically—

(11) The angle whose secant is 3.

(12) The angle whose sine is $\frac{1}{4}$.

(13) The angle whose tangent is $\frac{7}{24}$.

(14) The angle whose cotangent is $\frac{15}{8}$.

28. By means of the relations established above (§ 22), a great many identities can be proved. No special rules for working them can be given, except that, when other resources fail, it is best to express all ratios involved in terms of the sine and cosine.

Example 1.—Prove $\sin^6 A + \cos^6 A + 3\sin^2 A = 1 + 3\sin^4 A$.

$$\begin{aligned}\text{Sin}^6 A + \cos^6 A + 3\sin^2 A &= \sin^6 A + (1 - \sin^2 A)^3 + 3\sin^2 A \\ &= \sin^6 A + 1 - 3\sin^2 A + 3\sin^4 A - \sin^6 A + 3\sin^2 A \\ &= 1 + 3\sin^4 A.\end{aligned}$$

Example 2.—Show that $\operatorname{cosec}\theta - \cos\theta = \sqrt{\dfrac{1-\cos\theta}{1+\cos\theta}}$.

$$\begin{aligned}\operatorname{cosec}\theta - \cot\theta &= \frac{1}{\sin\theta} - \frac{\cos\theta}{\sin\theta} = \frac{1-\cos\theta}{\sin\theta} = \frac{1-\cos\theta}{\sqrt{1-\cos^2\theta}} \\ &= \sqrt{\frac{(1-\cos\theta)^2}{1-\cos^2\theta}} = \sqrt{\frac{1-\cos\theta}{1+\cos\theta}}.\end{aligned}$$

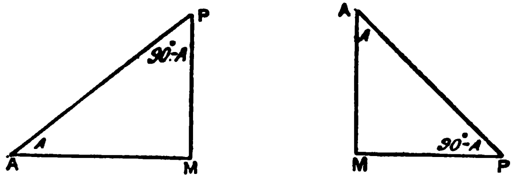
29. If A is any angle in degrees then 90° − A is called its complement.

To prove that

(1) The sine of any angle is equal to the cosine of its complement.

(2)	„	cosine	„	„	„	sine	„	„
(3)	„	tangent	„	„	„	cotangent	„	„
(4)	„	cotangent	„	„	„	tangent	„	„
(5)	„	cosecant	„	„	„	secant	„	„
(6)	„	secant	„	„	„	cosecant	„	„

Take a right-angled triangle P A M (Fig. 1). Let P A M = A, then A P M = 90° − A.



N.B.—Fig. 2 represents the same triangle placed on P M as base, so that the beginner can read off the trigonometrical ratios of 90° − A from it with greater ease.

We see that $\sin A = \frac{PM}{AP}$ (Fig. 1).

$$\cos (90^\circ - A = \frac{PM}{AP} \text{ (Fig. 2).}$$

$$\therefore \sin A = \cos (90^\circ - A)$$

So

$$\cos A = \frac{AM}{AP} = \sin (90^\circ - A)$$

$$\tan A = \frac{PM}{AM} = \cot (90^\circ - A)$$

$$\cot A = \frac{AM}{PM} = \tan (90^\circ - A)$$

$$\operatorname{cosec} A = \frac{AP}{PM} = \sec (90^\circ - A)$$

$$\sec A = \frac{AP}{AM} = \operatorname{cosec} (90^\circ - A).$$

These results are of great importance, and should be carefully remembered.

EXAMPLES III (*c*).

Prove the following identities :—

(1) $\tan A + \cot A = \sec A \operatorname{cosec} A$.

(2) $(\cos A + \sin A)(\operatorname{cosec} A - \sec A) = \cot A - \tan A$.

(3) $\operatorname{cosec} A + 2 \sec A - \cos A \cot A - 2 \sin A \tan A = \sin A + 2 \cos A.$

(4) $\cot A - \sec A \operatorname{cosec} A (1 - 2 \sin^2 A) = \tan A.$

(5) $(1 - 2 \cos^2 A)(\tan A + \cot A) = (\sin A - \cos A)(\sec A + \operatorname{cosec} A).$

(6) $\dfrac{\tan A + \tan B}{\cot A + \cot B} = \tan A \tan B.$

(7) $\sin^2 A \tan^2 A + \cos^2 A \cot^2 A = \tan^2 A + \cot^2 A - 1.$

(8) $\dfrac{\sin A + \sin B}{\cos A - \cos B} = \dfrac{\cos A + \cos B}{\sin B - \sin A}.$

(9) $\dfrac{\cot A \cos A}{\cot A - \cos A} = \dfrac{\cot A + \cos A}{\cot A \cos A}.$

(10) $\dfrac{\tan A + \sec A - 1}{1 + \tan A - \sec A} = \tan A + \sec A.$

(11) $\sin \theta (1 + \tan \theta) + \cos \theta (1 + \cot \theta) = \sec \theta + \operatorname{cosec} \theta.$

(12) $(1 + \sec A + \tan A)(1 + \operatorname{cosec} A + \cot A) = 2(1 + \tan A + \cot A + \sec A + \operatorname{cosec} A).$

(13) $(\sec \theta - \cos \theta)(\operatorname{cosec} \theta - \sin \theta) = \dfrac{\tan \theta}{1 + \tan^2 \theta}.$

(14) $(\tan \theta + \cot \theta - 1)(\sin \theta + \cos \theta) = \dfrac{\sec \theta}{\operatorname{cosec}^2 \theta} + \dfrac{\operatorname{cosec} \theta}{\sec^2 \theta}.$

(15) $\sin^2 A (1 + n \cot^2 A) + \cos^2 A (1 + n \tan^2 A) = \sin^2 A (n + \cot^2 A) + \cos^2 A (n + \tan^2 A).$

(16) $\sin^4 \theta + \cos^4 \theta = \sin^2 \theta (\operatorname{cosec}^2 \theta - 2 \cos^2 \theta).$

(17) $\sin^3 \theta + \cos^3 \theta = (1 - \sin \theta \cos \theta)(\sec \theta + \operatorname{cosec} \theta) \sin \theta \cos \theta.$

(18) $\sin^6 \theta - \cos^6 \theta = (2 \sin^2 \theta - 1)(1 - \sin \theta \cos \theta)(1 + \sin \theta \cos \theta).$

(19) $\dfrac{(\tan \theta - 1)(\operatorname{cosec} \theta - 1)}{\cot \theta - 1} + \dfrac{(\cot \theta + 1)}{(\tan \theta + 1)(\operatorname{cosec} \theta + 1)} = 0.$

(20) If $\cos \theta - \sin \theta = \sqrt{2} \sin \theta$ then $\cos \theta + \sin \theta = \sqrt{2} \cos \theta.$

CHAPTER IV.

VALUES OF THE RATIOS FOR CERTAIN ANGLES.

30. General Limitations.—Let P O M be any right-angled triangle. Let the angle P O M = A, and let P M O be the right angle.

Then whatever be the value of A, O P the side opposite the right angle can never be less than O M or P M.

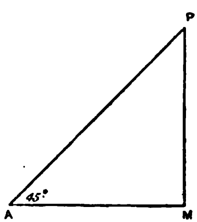
$$\therefore \frac{OM}{OP} \text{ and } \frac{PM}{OP} \text{ can never be greater than unity,}$$

$$\frac{OP}{OM} \text{ and } \frac{OP}{PM} \text{ can never be less than unity;}$$

∴ **cos A and sin A can never be greater than unity,** while sec A and cosec A can never be less than unity.

There is no limitation in the magnitudes of the tangents and cotangents of different angles which may assume any value from a very small one to a very large one.

31. To determine the Trigonometrical Ratios of an Angle of 45°.



Let P M A be a triangle right-angled at M.

Let P A M = 45°.

$$\therefore APM = 90° - 45° = 45°.$$

$$\therefore PM = AM. \quad \text{Euclid, I. 6.}$$

Let each of the equal sides P M and A M = a;

then $AP^2 = a^2 + a^2 = 2a^2$. Euclid, I. 47.

$$\therefore AP = \sqrt{2}.a;$$

$$\therefore \sin 45° = \frac{PM}{AP} = \frac{a}{a\sqrt{2}} = \frac{1}{\sqrt{2}},$$

$$\cos 45° = \frac{AM}{AP} = \frac{a}{a\sqrt{2}} = \frac{1}{\sqrt{2}},$$

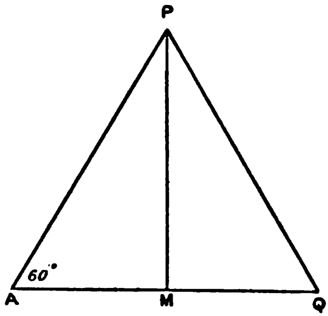
$$\tan 45^\circ = \frac{PM}{AM} = \frac{a}{a} = 1,$$

$$\operatorname{cosec} 45^\circ = \frac{1}{\sin 45^\circ} = \sqrt{2},$$

$$\sec 45^\circ = \frac{1}{\cos 45^\circ} = \sqrt{2},$$

$$\cot 45^\circ = \frac{1}{\tan 45^\circ} = 1.$$

32. To determine the Trigonometrical Ratios of an Angle of 60°.



Draw an equilateral triangle P A Q, each side being equal to 2 *a*.

Then each of the angles of the triangle will be equal to 60°.

From P draw P M perpendicular to A Q, then in the two triangles P A M, P M Q, the angles P A Q and P M A are equal to the angles P Q A and P M Q respectively, and P M is common;

∴ the triangle P A M = triangle P M Q, and A M = M Q;

$$\therefore AM = \frac{1}{2} AQ = a.$$

Then since $AP^2 = AM^2 + PM^2$,

or, $4a^2 = a^2 + PM^2$,

$$PM^2 = 3a^2,$$

$$\therefore PM = a\sqrt{3}.$$

$$\therefore \sin 60^\circ = \frac{PM}{AP} = \frac{a\sqrt{3}}{2a} = \frac{\sqrt{3}}{2}$$

$$\cos 60^\circ = \frac{AM}{AP} = \frac{a}{2a} = \frac{1}{2}$$

$$\tan 60^\circ = \frac{PM}{AM} = \frac{a\sqrt{3}}{a} = \sqrt{3}$$

$$\operatorname{cosec} 60^\circ = \frac{AP}{PM} = \frac{2a}{a\sqrt{3}} = \frac{2}{\sqrt{3}},$$

so

$$\sec 60^\circ = \frac{AP}{AM} = \frac{2a}{a} = 2$$

$$\cot 60^\circ = \frac{AM}{PM} = \frac{a}{a\sqrt{3}} = \frac{1}{\sqrt{3}}.$$

33. To determine the Trigonometrical Ratios of an Angle of 30°.

With figure as in the preceding Article, the angle A P M = 30°.

$$\therefore \sin 30^\circ = \frac{AM}{AP} = \frac{1}{2} \qquad \operatorname{cosec} 30^\circ = \frac{AP}{AM} = 2$$

$$\cos 30^\circ = \frac{PM}{AP} = \frac{\sqrt{3}}{2} \qquad \sec 30^\circ = \frac{AP}{QM} = \frac{2}{\sqrt{3}}$$

$$\tan 30^\circ = \frac{AM}{PM} = \frac{1}{\sqrt{3}} \qquad \cot 30^\circ = \frac{PM}{AM} = \sqrt{3}.$$

EXAMPLES IV (*a*).

If A = 60°, B = 45°, C = 30°, find the values of

(1) $\sin^2 A + \cos^2 C$.

(2) $\sin C + \cos^2 B$.

(3) $\tan B + \cot B$.

(4) $\cos B \sin B - \sin^2 C$.

(5) $\sec^2 A - \operatorname{cosec}^2 B + \cot^2 C$.

(6) $\dfrac{\sec C}{\tan C} - \dfrac{\sec B}{\cot A}$.

(7) $\dfrac{\sin A + \sin B + \sin C}{\cos A + \cos B + \cos C}$.

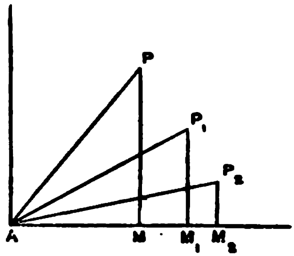
(8) $\dfrac{\tan A \tan B + \tan B \tan C + \tan C \tan A}{\tan A + \tan B + \tan C}$.

(9) $\dfrac{\sin A \cos B + \tan A \cos B}{\sin B \cos A + \cot A \cot C}$.

(10) $(\operatorname{cosec} A + \cot A)(\operatorname{cosec} C + \cot C) - 2(\sec A + \tan A)$.

34. To determine the Trigonometrical Ratios of an angle of 0°.

Let A P be a line of fixed length, and let the angle P A M become gradually smaller, by causing A P to revolve about A; let P_1, P_2 be two successive positions of P and draw $P_1 M_1$, $P_2 M_2$ perpendicular to M.



We notice (1) that P M is $> P_1 M_1$; $P_1 M_1 > P_2 M_2$; (2) that $AM_2 > AM_1$, $AM_1 > AM$.

∴ as the angle diminishes P M diminishes while A M increases.

In the limit when the angle P A M is zero P and M will coincide.

∴ $PM = 0$ and $AM = AP$.

$$\therefore \sin 0^\circ = \frac{PM}{AP} = 0, \cos 0^\circ = \frac{AM}{AP} = 1, \tan 0^\circ = \frac{PM}{AM} = 0,$$

$\sec 0^\circ = 1 \div 1 = 1$, $\operatorname{cosec} 0^\circ = 1 \div 0$, $\cot 0^\circ = 1 \div 0$.

The last two values require explanation. Let n be any quantity. Now let n become very large, then $\frac{1}{n}$ must become very small. Conversely the smaller $\frac{1}{n}$ is the larger n becomes. And in the limit we say that when $\frac{1}{n} = 0$, n is infinitely large: we then denote it by the symbol ∞.

∴ cosec 0° and cot 0° are both infinite and their value is denoted by ∞.

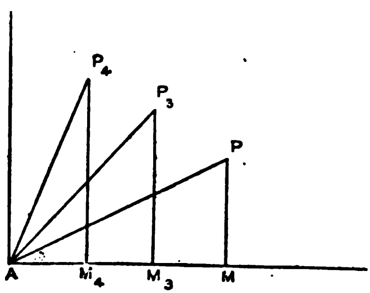
35. To determine the Trigonometrical Ratios of an Angle of 90°.

Let A P a line of fixed length revolve about A away from A M so that the angle P A M will become greater and greater.

Then we see

(1) that P_4M_4 is $> P_3M_3$, and P_3M_3 is $> PM$;

(2) that AM_4 is $< AM_3$, and $AM_3 < AM$.



$\therefore$ as the angle approaches 90°, P M becomes larger and A M becomes smaller. When the limit has been reached and A has become 90°, P M = A P and A M = 0.

We have therefore

$$\sin 90° = \frac{PM}{AP} = 1, \quad \cos 90° = \frac{AM}{AP} = 0,$$

$$\tan 90° = \frac{PM}{AM} = \infty, \quad \cot 90° = \frac{AM}{PM} = 0,$$

$$\operatorname{cosec} 90° = \frac{AP}{PM} = 1, \quad \sec 90° = \frac{AP}{AM} = \infty.$$

EXAMPLES IV (*b*).

If A = 90°, B = 60°, C = 45°, D = 30°, E = 0°, find the values of

(1) $\sin^2 A + \sin^2 B + \sin^2 C$.

(2) $\tan^2 C + \tan^2 D + \tan^2 E$.

(3) $\tan B \tan D - \tan C \tan E$.

(4) $\cot A \cot C + \cos B \cos E$.

(5) $\sec A \operatorname{cosec} B - \tan D \cot C$.

(6) $\dfrac{\sin A}{\cos B} + \dfrac{\tan C}{\cot D} + \dfrac{\sec E}{\operatorname{cosec} B}$.

(7) $\tan B \tan C \tan D$.

(8) $\operatorname{cosec} A \operatorname{cosec} C - \sec B \sec E$.

(9) $\dfrac{8 \cos^2 C - \tan^2 B}{\sin^2 B - \cot^2 A}$.

(10) $\dfrac{\cos E \operatorname{cosec} C}{\tan B \sec D}$.

36. The following table gives the results of this chapter, which should be committed to memory. (See § 29.)

Angle.	0°	30°	45°	60°	90°
Sine . . .	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
Cosine . .	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
Tangent . .	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞
Cosecant . .	∞	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1
Secant . .	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	∞
Cotangent .	∞	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0

CHAPTER V.

TRIGONOMETRICAL EQUATIONS.

37. In Trigonometrical Equations one or more angles are the unknown quantities which we have to find.

These angles usually appear represented by their trigonometrical ratios.

The solution consists in finding angles whose trigonometrical ratios, when substituted in the equation, satisfy it. For instance, in $\cos\theta + \sec\theta = \frac{5}{2}$, $\theta = \frac{\pi}{3}$ is a solution for $\cos\frac{\pi}{3} = \frac{1}{2}$, and $\sec\frac{\pi}{3} = 2$. $\therefore \cos\frac{\pi}{3} + \sec\frac{\pi}{3} = \frac{5}{2}$.

Example 1.—To solve

$$2\sin^2\theta - 2\sin\theta + 1 = 0 \quad . \quad . \quad . \quad .$$

Here only one trigonometrical ratio appears, namely, $\sin\theta$; we can therefore take it as our unknown quantity, as we would x in algebra, and solve the equation as a quadratic.

$$2\sin^2\theta - 3\sin\theta = -1$$

$$\sin^2\theta - \frac{3}{2}\sin\theta + \left(\frac{3}{4}\right)^2 = -\frac{1}{2} + \frac{9}{16};$$

$$\therefore \quad \sin\theta - \frac{3}{4} = \pm\frac{1}{4}.$$

$$\therefore \quad \sin\theta = 1 \text{ or } \frac{1}{2}.$$

$$\therefore \quad \theta = \frac{\pi}{2} \text{ or } \frac{\pi}{6}.$$

Example 2.—Solve

$$\sin^2\theta + \cos\theta + \frac{3}{\sec\theta} = \frac{11}{4}.$$

Before we can solve this equation we must reduce it to a form involving only one trigonometrical ratio of θ instead of three. A little thought will show that it will be most convenient to express $\sin^2\theta$ and $\sec\theta$ in terms of $\cos\theta$.

Since

$$\cos^2\theta + \sin^2\theta = 1, \quad \sin^2\theta = 1 - \cos^2\theta \quad . \quad . \quad . \quad (1)$$

and since $$\sec\theta = \frac{1}{\cos\theta}, \quad \frac{1}{\sec\theta} = \cos\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

substituting from (1) and (2) in the given equation,

we get $$1 - \cos^2\theta + \cos\theta + 3\cos\theta = \frac{11}{4},$$

whence $$\cos^2\theta - 4\cos\theta + \frac{7}{4} = 0;$$

$$\therefore \quad \cos^2\theta - 4\cos\theta + (2)^2 = -\frac{7}{4} + 4 = \frac{9}{4},$$

$$\cos\theta - 2 = \pm\frac{3}{2};$$

$\therefore \quad \cos\theta = \left[\frac{7}{2}\right]$ or $\frac{1}{2}$; but the cosine of an angle can never be greater than unity (§ 30), so the value in brackets is impossible; the other value gives $\theta = \frac{\pi}{3}$, which is a solution.

Example 3.—Solve

$$\left.\begin{aligned} \tan(A - B) &= \frac{1}{\sqrt{3}} \\ \sin(A + B) &= 1 \end{aligned}\right\} \begin{aligned} &. \quad . \quad . \quad . \quad . \quad . \quad (1) \\ &. \quad . \quad . \quad . \quad . \quad . \quad (2) \end{aligned}$$

This is an equation in which two unknown angles A and B have to be found.

Since $\tan 30^\circ = \frac{1}{\sqrt{3}}$, and since $\sin 90^\circ = 1$,

$$\left.\begin{aligned} A - B &= 30^\circ \\ A + B &= 90^\circ \end{aligned}\right\} \quad . \quad . \quad . \quad (3)$$

These are ordinary algebraical equations for finding A and B.

Adding them $\quad 2A = 120^\circ \quad \because \quad A = 60^\circ,$

subtracting one from the other $\quad 2B = 60^\circ \quad \therefore \quad B = 30^\circ$

Example 4.—Solve $\sin 3x = \cos 7x$.

We know that $\cos 7x = \sin(90^\circ - 7x)$ (§ 35);

$$\therefore \quad \sin 3x = \sin(90^\circ - 7x).$$

But equal angles have equal sines,

$$\therefore \quad 3x = 90^\circ - 7x \text{ is a solution,}$$

$$\therefore \quad 10x = 90^\circ,$$

$$\therefore \quad x = 9.$$

Example 5—

$$3 \sec \theta \operatorname{cosec} \theta = 2 \tan \theta + 2\sqrt{3},$$

$$\therefore \quad \frac{3}{\sin \theta \cos \theta} = \frac{2 \sin \theta}{\cos \theta} + 2\sqrt{3};$$

$$\therefore \quad 3 = 2 \sin^2 \theta + 2\sqrt{3} \sin \theta \cos \theta.$$

Divide both sides by $\cos^2 \theta$;

$$\therefore \quad \frac{3}{\cos^2 \theta} = 2 \tan^2 \theta + 2\sqrt{3} \tan \theta.$$

But $$\frac{3}{\cos^2 \theta} = 3 \sec^2 \theta = 3 (1 + \tan^2 \theta),$$

$$\therefore \quad 3 + 3 \tan^2 \theta = 2 \tan^2 \theta + 2\sqrt{3} \tan \theta,$$

$$\therefore \quad \tan^2 \theta - 2\sqrt{3} \tan \theta + 3 = 0,$$

$$\text{or} \quad (\tan \theta - \sqrt{3})^2 = 0,$$

$$\tan \theta - \sqrt{3} = 0,$$

$$\tan \theta = \sqrt{3};$$

$$\therefore \quad \theta = \frac{\pi}{3}.$$

38. Inverse Notation. If in solving an equation we obtain a result $\sin \theta = a$, we can construct the angle [§ 27, Ex. 4] without actually knowing the numerical value of θ, and say that it is "**an angle whose sine is a.**" This is written $\mathbf{\sin^{-1} a}$, which is merely an abbreviation. Therefore $\mathbf{\sin^{-1} a}$ represents an **angle**. Similarly, $\mathbf{\cos^{-1} m}$, $\mathbf{\tan^{-1} \frac{3}{4}}$ represent **angles**; $\cos^{-1} m$, an angle whose cosine is m, $\tan^{-1} \frac{3}{4}$ an angle whose tangent is $\frac{3}{4}$. This notation will be more fully dealt with in Chapter 27.

EXAMPLES V.

Solve the equations (neglecting negative values in the solution).

(1) $\dfrac{\sin\theta+1}{3}=\dfrac{3-2\sin\theta}{4}$.

(2) $2\cos\theta=\sqrt{2}$.

(3) $\dfrac{8}{3\tan\theta-1}=\dfrac{2}{\tan\theta-\frac{1}{2}}$.

(4) $\frac{1}{3}\left(\cot\theta+\frac{1}{2}\right)=\frac{1}{4}\left(\frac{2}{3}-\frac{\cot\theta}{5}\right)$.

(5) $2\sin^2\theta-5\sin\theta+2=0$.

(6) $2\sin\theta=\tan\theta$.

(7) $\tan 2\theta=1$.

(8) $\tan\theta+\cot\theta=2$.

(9) $\sin\theta=1-\cos\theta$.

(10) $3\sin\theta=2\cos^2\theta$.

(11) $\sec\theta\operatorname{cosec}\theta-\cot\theta=\sqrt{3}$.

(12) $\sec\theta\operatorname{cosec}\theta-\tan\theta=2$.

(13) $\sqrt{3}\sec^2\theta=4\tan\theta$.

(14) $\tan\theta+\sqrt{3}\cot\theta=1+\sqrt{3}$.

(15) $3\tan^4\theta-4\tan^2\theta+1=0$.

(16) $1+7\sin^2\theta=6\sin\theta\cos\theta$ (find $\tan\theta$).

(17) $\tan 4\theta=\cot 2\theta$.

(18) $\sin 5A=\cos 50^\circ$.

(19) $\begin{cases}\cos(2x-3y)=\dfrac{\sqrt{3}}{2}\\ \cos(3x-2y)=\dfrac{1}{2}\end{cases}$

(20) $\begin{cases}3\cos A+\sqrt{2}\cos B=2\frac{1}{2}\\ \dfrac{\cos A}{3}+\dfrac{\cos B}{\sqrt{2}}=\dfrac{2}{3}\end{cases}$

(21) $\begin{cases} \cos^2 A + \cos^2 B = \dfrac{5}{4} \\ \cos A \cos B = \dfrac{1}{2} \end{cases}$

(22) $\begin{cases} \sin A + \sin B = \dfrac{1+\sqrt{3}}{2} \\ \sin A \sin B = \dfrac{\sqrt{3}}{4} \end{cases}$

(23) $\dfrac{\sin A}{\sin B} = \sqrt{2}$, $\dfrac{\tan A}{\tan B} = \sqrt{3}$.

(24) $\cos\theta = \tan\phi$, $\cot\theta = \sin\phi$.

(25) Prove that

$$\sin^{-1}\frac{3}{5} = \cos^{-1}\frac{4}{5}$$

$$= \tan^{-1}\frac{3}{4}.$$

(26) Prove that

$$\tan^{-1}\frac{1}{2} = \operatorname{cosec}^{-1}\sqrt{5}.$$

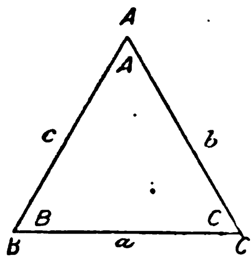
(27) Show that

$$\sin^{-1}\frac{33}{65} + \cos^{-1}\frac{56}{65} = 90°.$$

CHAPTER VI.

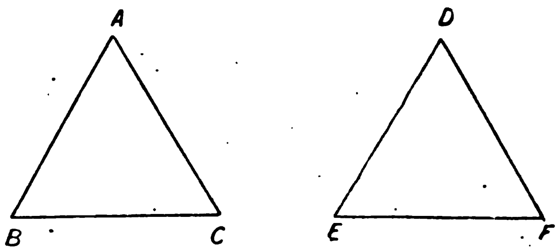
THE SOLUTION OF RIGHT-ANGLED TRIANGLES.

39. Let A B C be a triangle, then the sides B C, C A, A B and the angles B A C, C B A, A C B are called its parts. The sides in the above order are denoted by the letters *a*, *b*, *c*, the angles by A, B, C.



40. When certain of these parts are given we are enabled to find the magnitude of the remaining parts, and when we do this we are said to **solve the triangle.**

From the propositions which prove the equality of triangles in Euclid (I. 4, 8, 26) we see that when either two sides and an included angle; three sides; or one side and two angles (three parts in each case) are given, enough is given to determine all the other



parts of the triangle. For let A B C be a given triangle whose parts are known, make E F = B C, an angle D E F equal to B, and E D = B A and join D F.

Then by Euclid (I. 4)—

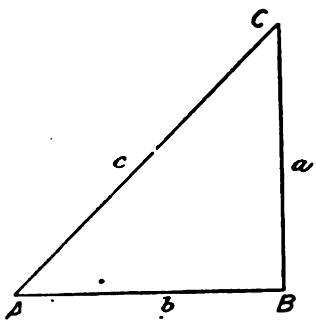
D F = A C, which is known;
$\angle$ F D E = A " "
$\angle$ D F E = C " "

Similar proofs will hold for the other cases.

41. In a right-angled triangle one part, the right angle, is always known, we, therefore require only two additional parts in order to determine the remaining parts.

Four different cases will occur.

42. (1) Given a and b, to find A, B, c; C being the right angle.



Since C is 90°,

$$\therefore \tan A = \frac{a}{b}, \quad \therefore A \text{ is known};$$

$$\text{but } B = 90° - A, \quad \therefore B \text{ is known.}$$

$$\frac{c}{b} = \sec A, \quad \therefore c = b \sec A,$$

$$\therefore c \text{ is known.}$$

Example.—Let $a = 12$,

$$b = 4\sqrt{3},$$

$$\tan A = \frac{12}{4\sqrt{3}} = \sqrt{3},$$

$$\therefore A = 60°$$

$$B = 90° - 60° = 30°$$

$$c = 4\sqrt{3} \sec 60° = 8\sqrt{3}.$$

43. (2) Given c and b, we have to find A, B and a.

$$\text{Since C is } 90°, \cos A = \frac{b}{c}, \quad \therefore A \text{ is known,}$$

$$\therefore (90° - A), \text{ or B, is known,}$$

$$\text{and } a = c \sin A, \quad \therefore a \text{ is known.}$$

Similarly if c and a were given we could find A, B, and b.

Example.—Given $c = 2b$ and $b = 100$ yards, to solve the triangle.

$$\cos A = \frac{b}{c} = \frac{1}{2}, \quad \therefore A = 60°$$

$$B = 90° - 60° = 30°,$$

$$a = c \sin 60° = 200 \cdot \frac{\sqrt{3}}{2} \text{ yds.}$$

$$= 173 \cdot 2 \text{ yds.}$$

44. (3) Let a and A be given, we have to find B, b, c.

$$B = 90° - A,$$
$$b = a \cot A, \text{ which is known,}$$
$$c = a \operatorname{cosec} A, \quad ,, \quad ,,$$

Similar solutions can be found if a and B or b and A or b and B are given.

Example.—Let $a = 10$, $B = 45°$ to find A, b, c.

$$A = 90° - 45° = 45°,$$
$$b = a \tan B = 10 \times \tan 45° = 10,$$
$$c = a \sec B = 10 \times \sec 45° = 10 \sqrt{2} = 14 \cdot 4.$$

45. (4) Let c and A be given; it is required to find B, a, b.

$$B = 90° - A, \text{ which is known,}$$
$$a = c \sin A, \quad ,, \quad ,,$$
$$b = c \cos A, \quad ,, \quad ,,$$

A similar solution holds if c and B are given.

Example.—The hypothenuse of a right-angled triangle is 30 inches long; one of the acute angles is known to be 30°; it is required to find the sides and the other angle.

The other angle is $90° - 30° = 60°$.

The sides are found from

$$a = 30 \sin 30° = 15,$$
$$b = 30 \cos 30° = 30 \times \frac{\sqrt{3}}{2} = 15 \sqrt{3}.$$

46. We may be given one side and the ratio of the other two, as in the following example.

Example.—Solve a right-angled triangle when

$$\frac{b}{a} = \frac{45}{28} \text{ and } c = 10 \cdot 6.$$

We have $$\frac{b}{a} = \frac{45}{28} = k \text{ suppose;}$$

$$\therefore \quad b = 45\,k \text{ and } a = 28\,k;$$

but $$c^2 = b^2 + a^2;$$

that is $$112 \cdot 36 = 2025\,k^2 + 784\,k^2$$
$$= 2809\,k^2.$$

Whence
$$k^2 = \frac{112 \cdot 36}{2809} = \frac{1}{25}$$

$$k = \frac{1}{5},$$

$$\therefore \quad b = 9 \text{ and } a \doteq 5 \cdot 6.$$

With our present knowledge we cannot find the values of A and B, but we may write

$$\tan A = \frac{28}{45}, \quad \text{or,} \quad A = \tan^{-1}\frac{28}{45};$$

and similarly
$$B = \tan^{-1}\frac{45}{28}.$$

EXAMPLES VI.

Find the other parts of the triangle in the following cases, C being the right angle, and a, b, c the sides opposite A, B, and C respectively.

(Assume $\sqrt{2} = 1 \cdot 4$, $\sqrt{3} = 1 \cdot 73$.)

(1) $a = 50$, $c = 70$.

(2) $a = 6$, $b = 6$.

(3) $b = 5$, $A = 60°$.

(4) $b = 5$, $A = 45°$.

(5) $c = 30$, $B = 30°$.

(6) $c = 30$, $b = 15$.

(7) $b = 173$, $B = 60°$.

(8) $c = 14$, $A = 45°$.

(9) $c = 40$, $b = 34 \cdot 6$.

(10) $a = 12$, $\frac{c}{b} = \frac{2}{\sqrt{3}}$.

(11) $\sin A = \frac{1}{2}$, $a = 10$.

(12) $\tan A = 1$, $c = 28$.

(13) $\sec A = 2$, $b = 10$.

Find the sides only, when

(14) $B = \tan^{-1}\frac{3}{4}$, $c = 25$.

(15) $A = \cos^{-1}\frac{5}{13}$, $a = 24$.

(16) $A = \sin^{-1}\frac{99}{101}$, $b = 10$.

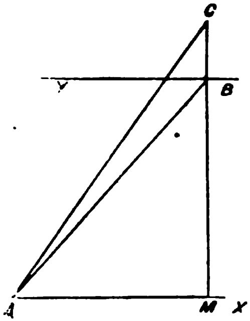
(17) $\frac{a}{b} = \frac{11}{60}$, $c = 122$.

(18) $\frac{c}{b} = \frac{37}{12}$, $a = 7$.

CHAPTER VII.

EASY PROBLEMS IN THE MEASUREMENT OF HEIGHTS AND DISTANCES.

47. Let B C denote the length of some vertical inaccessible object, such as a flag-staff on the top of a tower, of height B M.



Suppose an observer stationed at A, and let A X be a horizontal straight line through M, the foot of the tower. Draw B Y horizontal through B, and in the same vertical plane as A B. Then the angle B A M, made by B A with the horizontal line A X, is called the *angle of elevation* of B with respect to A; the angle Y B A is called the *angle of depression* of A with respect to B. Since B Y is parallel to A M, the angles Y B A, B A M, being alternate angles, are equal. The angle C A M is similarly the angle of elevation of C with respect to A. The angle C A B, which is the difference of the angles of elevation at A of the points B and C, is called the angle subtended by B C at A.

The angles of elevation and depression of objects are measured with instruments, an account of which is found in Treatises on Surveying.

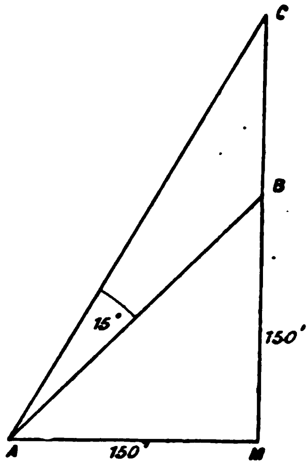
By means of such angles, and of lengths measured on level ground, the heights of inaccessible objects and their distances from the observer may be found.

The reader ought to master the methods adopted in the worked-out examples, and then work out many questions for himself.

In order not to confuse the beginner, the point of observation has been taken in the horizontal plane in the first two examples.

Example 1.—The spire on the top of a tower 150 ft. high subtends an angle of 15° at a point 150 ft. from the base of the tower, measured along a horizontal plane. Find the height of the spire.

Let B M represent the height of the tower, A M the length measured along the horizontal plane, B C the height of the spire. Then the angle B A C = 15° by hypothesis.



Now

$$\tan BAM = \frac{BM}{AM} = \frac{150}{150} = 1.$$

$$\therefore \angle BAM = 45°.$$

$$\therefore \angle CAM = 45° + 15° = 60°.$$

Again,

$$\frac{CM}{AM} = \tan CAM = \tan 60° = \sqrt{3}.$$

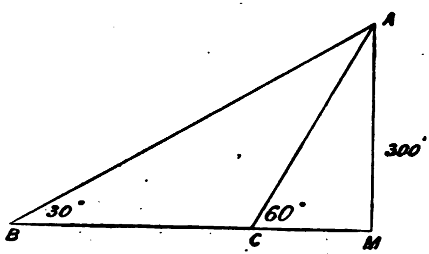
$$\therefore CM = AM\sqrt{3}.$$

Or, $BC + 150 = 150\sqrt{3}$.

$$\therefore BC = 150(\sqrt{3} - 1) = 150 \times \cdot 732$$

$$= 109 \cdot 8 \text{ ft.}$$

Example 2.—From the top of a cliff 300 ft. high the angles of depression of two ships in the same vertical plane as the observer are found to be 60° and 30°. Find the distance between them.



Let A be the station of the observer, B and C the ships. It is required to find B C.

$$\frac{BM}{AM} = \cot 30° = \sqrt{3}. \quad \therefore BM = 300\sqrt{3} \quad . \; . \; . \quad (1)$$

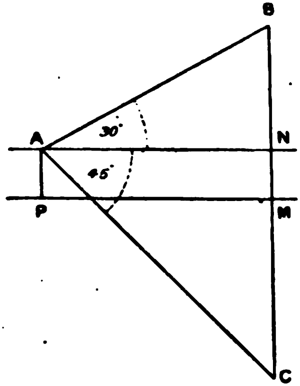
$$\frac{CM}{AM} = \cot 60° = \frac{1}{\sqrt{3}}. \quad \therefore CM = \frac{300}{\sqrt{3}} \quad . \; . \; . \; . \; . \quad (2)$$

subtracting (2) from (1), $BC = 300\left(\sqrt{3} - \frac{1}{\sqrt{3}}\right)$

$$= 200\sqrt{3}$$

$$= 346 \cdot 4 \text{ nearly.}$$

Example 3.—A man, whose eye is 5 ft. 6 in. above the ground, observes the angle of elevation of the top of a flag-staff to be 30°, and the angle of depression of the same point, as seen reflected in a pool of water, to be 45°. Find the height of the flagstaff.



Let A be the position of the eye of the observer, P M the horizontal plane, B M the height of the flag-staff, C M its reflexion.

Then $NM = AP = 5\frac{1}{2}$;

$$\frac{AN}{BN} = \cot 30° = \sqrt{3} \quad . \quad . \quad . \quad . \quad (1)$$

$$\frac{AN}{CN} = \cot 45° = 1 \quad . \quad . \quad . \quad . \quad . \quad (2)$$

dividing (1) by (2), and remembering that $CM = BM$,

$$\frac{CN}{BN} = \sqrt{3}, \quad \text{or,} \quad \frac{BM + 5\frac{1}{2}}{BM - 5\frac{1}{2}} = \sqrt{3}.$$

$$\therefore \quad BM = \frac{11}{2} \times \frac{\sqrt{3}+1}{\sqrt{3}-1} = \frac{11}{2}(2+\sqrt{3}) = 20{\cdot}526 \text{ ft. nearly.}$$

EXAMPLES VII (*a*).

(Assume $\sqrt{2} = 1{\cdot}4$ and $\sqrt{3} = 1{\cdot}73$.)

(1) What is the altitude of the sun when the shadow of a vertical stick is the same length as the stick?

(2) A man walks on level ground 100 ft. from the foot of a tower and finds that the angle of elevation of the top of the tower is then 60°. Find the height of the tower.

(3) A kite is flying with 140 yds. of string let out. Supposing the string to be fully stretched and at an angle of 45° with the horizontal, what is the height of the kite?

(4) At a point 240 ft. from the foot of a tower the angle of elevation of the top of the tower is 30°, and of the top of its steeple is 45°. Find the height of the steeple.

(5) What is the height of a lamp post when a vertical stick 3 ft. long and 12 ft. from the post casts a shadow 4 ft. long?

(6) The straight line joining the eye of an observer to a bird makes an angle of 60° with the horizon; after the bird has flown 100 yards horizontally away from the observer, the angle is only 30°. At what distance is the bird in the second position from the observer?

(7) Find the perpendicular height of a mountain whose summit, $2\frac{1}{4}$ miles distant, has an elevation of 9° 18′. (Given sin 9° 18′ = ·1616.)

(8) A ladder 39 ft. long reaches to a window 36 feet from the ground; find the distance of its foot from the wall: also find the length of a support one end of which is nailed at right angles to the ladder and the other rests against the bottom of the wall.

(9) The angle of elevation of a house on the bank of a river, observed from the opposite side, is 45°. The observer walks back 35 ft. and finds the angle of elevation is then 30°. Find the height of the house and breadth of the river.

(10) A man walking along a straight road observes from one milestone a tower in a direction making an angle of 30° with that of the road; at the next milestone the angle is 45°. How far is the tower from the nearest point on the road, and how far is the first milestone from the same point?

(11) A path goes for 318 ft. straight up a hill at an angle of 30°, and is followed by a flight of steps in the same direction 200 ft. in length at an angle of 45° leading to the top. Show that the height of the hill is nearly 300 feet.

(12) Two observers, one mile apart, observe a balloon in the vertical plane passing through them both at elevations of $22\frac{1}{2}°$ and $67\frac{1}{2}°$ respectively. If $\tan 22\frac{1}{2}° = \frac{2}{5}$ find the height of the balloon above the ground.

(13) A man on a cliff observes a boat at an angle of depression of 30° making for the shore immediately beneath him. Three minutes later the angle of depression of the boat is 60°. How soon will it reach the shore?

(14) The angles of elevation of the top of a tower at distances of 100 and 225 ft. from its base are found to be complementary. Find the height of the tower.

(15) At a distance of 300 ft. from the base of a tower 150 ft. high the angle of elevation of a flagstaff on it is 30°. Find the height of the flagstaff.

(16) From the top of a hill the angles of depression of the top and bottom of a house 30 ft. high are found to be 45° and $\tan^{-1}\frac{8}{7}$. Find the height of the hill above the ground on which the house stands.

(17) A person travelling southwards observes two objects in the S.E. After 8 miles travelling, one of them is N.E. and the other E. Show that their distances from him are then $4\sqrt{2}$ miles and 8 miles respectively.

(18) A ship sails S.S.E. for 3 miles, and E.S.E. for 2 more. How far east is it of its starting point? Given $\tan 22\frac{1}{2}° = \cdot 4$.

(19) From the top of a tower 108 ft. high, the angle of depression of the top and bottom of a column on the same horizontal plane are 30° and 60° respectively. Find the height of the column.

(20) A man sailing due west observes two fixed objects directly north: after sailing 6 miles the directions of the objects make angles 60°, 30° with the ship's course. Find how far apart the objects are.

48. In the following examples the measurements are not necessarily all in the same plane. The actual work is not harder, but more care is required in drawing the figures so as to avoid confusion.

Example 1.—A man at A (Fig. 1) observes the angle of elevation of the top of a flagstaff B C to be 45°. He then walks 100 ft. to D in a direction perpendicular to A B.

At D he finds the angle of elevation of C to be 30°. Find the height of the flagstaff.

Join D C and D B.

Then $\quad A\,B = B\,C \cot 45° = B\,C$

$$B\,D = B\,C \cot 30° = B\,C\,\sqrt{3},$$

but since the angle B A D = 90°

$$B\,D^2 = A\,B^2 + A\,D^2;$$

or, $\quad 3\,B\,C^2 = B\,C^2 + 100^2.$

$\therefore \quad 2\,B\,C^2 = 100^2.$

$$B\,C = \frac{100}{\sqrt{2}} = 50\,\sqrt{2} = 70 \cdot 7 \text{ ft.}$$

Fig. 1.

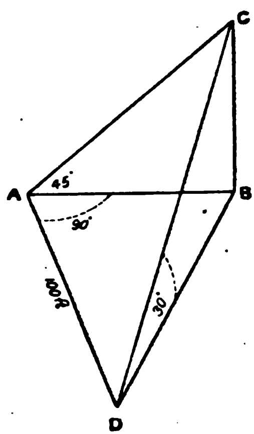
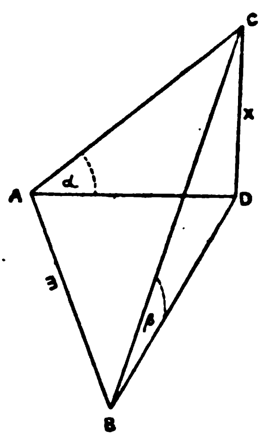


Fig. 2.



Example 2.—A, placing himself due west of a mountain, observes the angle of elevation of its summit to be α, B stands due south of it at a place m miles distant from A and finds the angle of elevation of the summit at this point to be β. Show that the mountain is $\dfrac{5280\,m}{\sqrt{\cot^2 \alpha + \cot^2 \beta}}$ ft. high. (See Fig. 2.)

Let C denote the summit of the mountain, D the point on the horizontal plane vertically below it, draw A D perpendicular to C D and B D perpendicular to the plane A D C containing the lines A D and C D, *i.e.* perpendicular to the plane of the paper.

(8) If $\tan\theta = \frac{2}{3}$, find $\sin\theta$ and $\cos\theta$ and vers θ.

(9) Prove that $\tan^2 60° - 2\tan 45° = \cot^2 30° - 2\sin^2 30° - \frac{3}{4}\operatorname{cosec}^2 45°$.

(10) If $\sin\alpha\cos\alpha = \frac{\sqrt{3}}{4}$, find two values of α.

(11) If the circular measure of an angle be $\cdot 031416$, find its English and French measures (assume $\pi = 3\cdot 1416$).

(12) Prove that $\sin^2\frac{\pi}{3} - \cos^2\frac{\pi}{4} = \frac{3}{4}\tan^2\frac{\pi}{6}$.

(13) Find the number of French minutes in a degree, and also in $67° 27' 12''$.

(14) The difference of the circular measure of two angles is $\frac{\pi}{10}$, and the ratio of the numbers of degrees in one to the other is $\frac{3}{2}$. Find the angles.

(15) Prove that $\sec^2\theta\tan\theta + 2\sec\theta\operatorname{cosec}\theta + \operatorname{cosec}^2\theta\cot\theta = \sec^3\theta\operatorname{cosec}^3\theta$. What is the value of each side of the expression when $\theta = \frac{\pi}{4}$?

(16) Find the number of degrees in the angle at the centre of a circle whose radius is 25 ft. which is subtended by an arc of 30 ft. $\left(\text{assume } \pi = \frac{22}{7}\right)$.

(17) From two places half a mile apart; at the same sea-level and in the same vertical plane with the top of a mountain, the angles of elevation of the latter are seen to be 45° and 35°. Given $\tan 35° = \cdot 7$, find the height of the mountain.

(18) Solve the equation, $2\cos\theta = 3 - \sec\theta$.

(19) Construct the angle whose secant is $\sqrt{5}$.

(20) What is, roughly, the length of a flag which extended to its full length at the distance of half a mile subtends an angle of 35′ at the eye? (see § 14.)

(21) Solve the equation, $3\sec^2\theta = 2\operatorname{cosec}\theta$.

(22) A vertical staff throws a shadow of 20 ft. when the elevation of the sun above the horizon is 63° 26′. Find the height of the staff, knowing that $\cos 63° 26' = \frac{1}{\sqrt{5}}$.

(23) Prove that $1 - \tan^2 A \tan^2 B = \dfrac{\sin^2 B - \cos^2 A}{\cos^2 A \cos^2 B}$.

(24) Express in degrees, &c., the angle whose circular measure is $\dfrac{7\pi}{32}$.

(25) The sine of the least angle of a triangle is $\dfrac{1}{2}$, and the difference between the other two angles is $7° 30'$. Find all the angles.

(26) Prove that $\dfrac{\operatorname{cosec} A - \cot A}{\sec A + \tan A} = \dfrac{\sec A - \tan A}{\operatorname{cosec} A + \cot A}$.

(27) Show that to turn circular measure into seconds we must multiply by 206265, and to turn seconds into circular measure by ·0000048.

(28) The tangent of an angle is $\dfrac{1}{2}$; find the other trigonometrical ratios.

(29) Solve the equations $\begin{cases} \cos^2 A + \cos^2 B = \dfrac{344}{225}. \\ \sin A \sin B = \dfrac{1}{5}. \end{cases}$

(30) A, B, C, D are four points in a straight line on a horizontal plane. At A and C are flagstaffs, which subtend angles of 60° and 30° respectively at B, and whose tops are in the same straight line with D. $AB = 100$, $CD = 40$. Find BC.

(31) When $A = 60°$, $B = 45°$, $C = 30°$, find the value of

$$\frac{\sin B \cos B + \sec A \operatorname{cosec} C - \tan A \tan C}{\cot A + \cot C + \sqrt{3} \cot B}.$$

(32) A man at a point A wishes to walk to a point B, four miles distant. In his way is a circular lake, the radius of which is one mile, the centre being midway between A and B. How far at least will the man have to walk in order to arrive at B?

(33) Express $23^g\ 6'\ 25''$ in degrees, &c.

(34) If $\tan A = \dfrac{1 + x}{1 - x}$, find $\sin A$ and $\sec A$.

(35) Show that $\dfrac{\sin 45° - \sin 30°}{\sin 45° + \sin 30°} = (\sec 45° - \tan 45°)^2$.

(36) Show that the sum of the tangent and cotangent of the same angle is never less than 2.

(37) Solve the equation, $6 \cot^2 \theta - 4 \cos^2 \theta = 1$.

(38) A man looking north observes that the edge of a flag known to be a mile off subtends an angle of 10′ at his eye; the wind being south-west. Find approximately the length of the flag. (Take $\pi = \frac{22}{7}$; $\sqrt{2} = 1 \cdot 4$.)

(39) Construct an angle whose tangent is $\sqrt{7}$.

(40) A circular reservoir subtends at a certain point on the ground an angle of 60°; at a point 100 ft. nearer it subtends an angle of 90°. Find its diameter correct to a foot.

PART II.

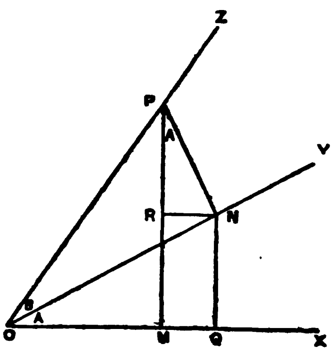
CHAPTER VIII.

TRIGONOMETRICAL RATIOS OF THE SUM AND DIFFERENCE OF TWO ANGLES.

49. To express the sine, cosine, and tangent of the sum of two angles in terms of trigonometrical ratios of these angles.

Let A and B be two angles whose sum is less than 90°; then we shall show that

$$\sin(A+B) = \sin A \cos B + \cos A \sin B \quad . \; . \; . \quad (\alpha)$$
$$\cos(A+B) = \cos A \cos B - \sin A \sin B \quad . \; . \; . \quad (\beta)$$
$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B} \quad . \; . \; . \quad (\gamma).$$



Let the angles X O Y (A) and Y O Z (B) be taken adjacent to each other as in the figure. Take any point P in O Z; draw P M, P N perpendicular to O X and O Y respectively; draw N R parallel to O X, N Q parallel to P M.

Then $\angle RPN = 90° - \angle PNR = \angle RNO = \angle NOQ$

$$\therefore \angle RPN = A.$$

$$(\alpha)\ \text{Sin}\ (A + B) = \sin POM = \frac{PM}{OP} = \frac{RM + PR}{OP}$$

$$= \frac{NQ}{OP} + \frac{PR}{OP}$$

$$= \frac{NQ}{ON} \cdot \frac{ON}{OP} + \frac{PR}{PN} \cdot \frac{PN}{OP}$$

$$= \sin A \cos B + \cos A \sin B.$$

$$(\beta)\ \cos (A + B) = \cos POM = \frac{OM}{OP} = \frac{OQ - MQ}{OP}$$

$$= \frac{OQ}{OP} - \frac{RN}{OP}.$$

$$= \frac{OQ}{ON} \cdot \frac{ON}{OP} - \frac{RN}{PN} \cdot \frac{PN}{OP}$$

$$= \cos A \cos B - \sin A \sin B.$$

$$(\gamma)\ \tan (A + B) = \tan POM = \frac{PM}{OM}$$

$$= \frac{RM + PR}{OQ - MQ} = \frac{NQ + PR}{OQ - RN}$$

$$= \frac{\frac{NQ}{OQ} + \frac{PR}{OQ}}{1 - \frac{RN}{OQ}} = \frac{\frac{NQ}{OQ} + \frac{PR}{OQ}}{1 - \frac{RN}{NQ} \cdot \frac{NQ}{OQ}}$$

$$= \frac{\tan A + \frac{PR}{OQ}}{1 - \tan A \cdot \frac{RN}{NQ}}.$$

Now since the triangles PRN, NOQ, are similar, the sides opposite equal angles in each triangle bear each other an equal ratio,

that is $$\frac{PR}{OQ} = \frac{RN}{NQ} = \frac{PN}{ON} = \tan B.$$

$$\therefore \tan (A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}.$$

50. The last of these three results may be more easily deduced from the other two as follows:

$$\text{Tan}\,(A+B)=\frac{\sin(A+B)}{\cos(A+B)}=\frac{\sin A\cos B+\cos A\sin B}{\cos A\cos B-\sin A\sin B}$$

$$=\frac{\dfrac{\sin A\cos B}{\cos A\cos B}+\dfrac{\cos A\sin B}{\cos A\cos B}}{1-\dfrac{\sin A\sin B}{\cos A\cos B}}$$

(dividing numerator and denominator by $\cos A\cos B$)

$$=\frac{\tan A+\tan B}{1-\tan A\tan B}.$$

51. *Example* 1.—

If $\tan A=\frac{1}{2}$, $\tan B=\frac{1}{3}$, show that $A+B$ is 45°.

For $$\tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B}=\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{6}}=1$$

but $\tan 45^\circ=1$. $\therefore$ 45° is a value of $A+B$.

Note.—It will be shown in Part III. that $A+B$ has other values as well; these, however, are greater than 90°.

Example 2.—Find $\sin(A+B)$ when $\sin A=\frac{3}{5}$ and $\tan B=\frac{5}{12}$.

Since $\sin(A+B)=\sin A\cos B+\cos A\sin B$, we require to find from our data $\cos B$, $\sin B$ and $\cos A$.

$$\cos A=\sqrt{1-\sin^2 A}=\frac{4}{5}$$

$$\cos B=\frac{1}{\sqrt{1+\tan^2 B}}=\frac{12}{13},$$

$$\sin B=\tan B\times\cos B=\frac{5}{12}\times\frac{12}{13}=\frac{5}{13}$$

$$\therefore\ \sin(A+B)=\frac{3}{5}\times\frac{12}{13}+\frac{4}{5}\times\frac{5}{13}=\frac{56}{65}.$$

Example 3.—To find the value of the sine of an angle of 75°.

Since $30° + 45° = 75°$

$$\therefore \sin 75° = \sin(30° + 45°) = \sin 30° \cos 45° + \cos 30° \sin 45°$$

$$= \frac{1}{2} \cdot \frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{2} \cdot \frac{1}{\sqrt{2}} = \frac{\sqrt{3}+1}{2\sqrt{2}}.$$

Similarly cos 75° and tan 75° can be found, and the values of the other ratios are at once obtained by applying § 17.

EXAMPLES VIII (*a*).

(1) Find the value of cos 75°.

(2) Find the value of tan 75°.

(3) Write down the values of cot 75°, sec 75°, and cosec 75°.

Find the values of

(4) $\cos(A + B)$ when $\cos A = \frac{4}{5}$ and $\sin B = \frac{5}{13}$.

(5) $\sin(A + B)$ when $\tan B = \frac{3}{4}$ and $\cos A = \frac{12}{13}$.

(6) $\tan(A + B)$ when $\sin A = \frac{3}{5}$ and $\tan B = \frac{5}{12}$.

(7) $\sin(A + B)$ when $\tan A = \frac{11}{60}$ and $\cos B = \frac{3}{5}$.

(8) $\cot(A + B)$ when $\cos A = \frac{24}{25}$ and $\cot B = \frac{4}{3}$.

(9) $\sec(A + B)$ when $\sin A = \frac{11}{61}$ and $\sin B = \frac{7}{25}$.

Prove the following identities:—

(10) $\dfrac{\cos(A + B)}{\cos A \sin B} = \cot B - \tan A.$

(11) $\dfrac{\sin(A + B)}{\sin A \sin B} = \cot B + \cot A.$

(12) $\cot(A + B) = \dfrac{\cot A \cot B - 1}{\cot A + \cot B}.$

(13) $\sin(A + B) \sec A \sec B = \tan A + \tan B.$

(14) $\tan(A + B) = \dfrac{1 + \cot A \tan B}{\cot A - \tan B}$.

(15) $\tan(45° + A) = \dfrac{1 + \tan A}{1 - \tan A}$.

(16) $2 \sin(45° + A) = \sqrt{2}(\sin A + \cos A)$.

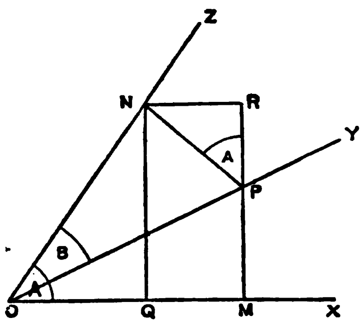
(17) $\dfrac{\cos(A + B) \cos C}{\cos(A + C) \cos B} = \dfrac{1 - \tan A \tan B}{1 - \tan A \tan C}$.

(18) $\dfrac{\sin(B + C) \cos A}{\sin(C + A) \cos B} = \dfrac{\tan B + \tan C}{\tan C + \tan A}$.

(19) $\cos(A + B) \sin C + \cos(C + A) \sin B = \cos A \sin(B + C) - 2 \sin A \sin B \sin C$.

(20) $\cos(A + B) \sin B - \cos(A + C) \sin C = \sin(A + B) \cos B - \sin(A + C) \cos C$.

52. To express the sine, cosine, and tangent of the difference of two angles in terms of trigonometrical ratios of these angles.



Let A and B be two angles each $< 90°$ of which A is the greater, we shall show that

$$\sin(A - B) = \sin A \cos B - \cos A \sin B \quad . \; . \; (\delta)$$
$$\cos(A - B) = \cos A \cos B + \sin A \sin B \quad . \; . \; (\epsilon)$$
$$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B} \quad . \; . \; . \; . \; . \; . \; (\zeta)$$

Let the angles X O Z (A) and Y O Z (B) be taken as in the figure so that O Y falls within the angle X O Z, then the angle

$X O Y = A - B$. Take any point P in O Y and draw P M and P N perpendicular to O X and O Z respectively. Through N draw N Q parallel to P M and N R parallel to O X meeting M P produced in R.

Then $\angle N P R = P N Q =$ complement of $O N Q = N O Q = A$.

$$\therefore\ (\delta)\ \sin(A - B) = \sin POM = \frac{PM}{OP} = \frac{RM - RP}{OP}.$$

$$= \frac{NQ}{OP} - \frac{RP}{OP} = \frac{NQ}{ON} \cdot \frac{ON}{OP} - \frac{RP}{PN} \cdot \frac{PN}{OP}$$

$$= \sin A \cos B - \cos A \sin B.$$

$$(\epsilon)\ \cos(A - B) = \cos POM = \frac{OM}{OP} = \frac{OQ + QM}{OP}$$

$$= \frac{OQ}{OP} + \frac{NR}{OP}$$

$$= \frac{OQ . ON}{ON . OP} + \frac{NR . PN}{PN . OP}$$

$$= \cos A \cos B + \sin A \sin B.$$

$$(\zeta)\ \tan(A - B) = \tan POM = \frac{PM}{OM} = \frac{RM - RP}{OQ + QM}$$

$$= \frac{NQ - RP}{OQ + RN} = \frac{\dfrac{NQ}{OQ} - \dfrac{RP}{OQ}}{1 + \dfrac{RN}{OQ}}$$

$$= \frac{\dfrac{NQ}{OQ} - \dfrac{RP}{OQ}}{1 + \dfrac{RN}{NQ} \cdot \dfrac{NQ}{OQ}} = \frac{\tan A - \dfrac{RP}{OQ}}{1 + \tan A \cdot \dfrac{RN}{NQ}}.$$

But since the triangles N P R and N O Q are similar, the sides opposite to the equal angles in each triangle bear to each other an equal ratio.

$$\therefore\ \frac{RP}{OQ} = \frac{RN}{NQ} = \frac{PN}{ON} = \tan B,$$

$$\therefore\ \tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}.$$

This result may be deduced from the two former in the same way as in § 50.

53. *Example* 1.—To find the trigonometrical ratios of an angle of 15°.

$$45^\circ - 30^\circ = 15^\circ$$

$$\therefore \sin 15^\circ = \sin(45^\circ - 30^\circ) = \sin 45^\circ \cos 30^\circ - \cos 45^\circ \sin 30^\circ$$

$$= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \cdot \frac{1}{2} = \frac{\sqrt{3}-1}{2\sqrt{2}}.$$

Similarly the values of the cosine and tangent may be found. The cosecant, secant, and cotangent of 15° may be obtained by inverting the sine, cosine, and tangent respectively.

Example 2.—A, B, C, being angles in descending order of magnitude, show that—

$$\frac{\sin(A-B)}{\cos A \cos B} + \frac{\sin(B-C)}{\cos B \cos C} - \frac{\sin(A-C)}{\cos A \cos C} = 0$$

$$\frac{\sin(A-B)}{\cos A \cos B} = \frac{\sin A \cos B - \cos A \sin B}{\cos A \cos B} = \frac{\sin A}{\cos A} - \frac{\sin B}{\cos B}$$

$$= \tan A - \tan B;$$

similarly

$$\frac{\sin(B-C)}{\cos B \cos C} = \tan B - \tan C, \quad \frac{\sin(A-C)}{\cos A \cos C} = \tan A - \tan C,$$

$$\therefore \frac{\sin(A-B)}{\cos A \cos B} + \frac{\sin(B-C)}{\cos B \cos C} - \frac{\sin(A-C)}{\cos A \cos C}$$

$$= \tan A - \tan B + \tan B - \tan C + \tan C - \tan A = 0.$$

EXAMPLES VIII (*b*).

(1) Find the value of cos 15°.

(2) Find the value of tan 15°.

(3) Write down the values of cot 15°, sec 15°, and cosec 15°.

Find the value of

(4) $\sin(A - B)$ when $\sin A = \frac{12}{13}$ and $\cos B = \frac{4}{5}$.

Find the value of

(5) $\tan(A - B)$ under the same conditions.

(6) $\cos(A - B)$ when $\tan A = \frac{4}{3}$ and $\sin B = \frac{7}{25}$.

(7) $\operatorname{cosec}(A - B)$ when $\sin A = \frac{24}{25}$ and $\sin B = \frac{11}{61}$.

(8) $\cos(A - B)$ when $\cos A = \frac{3}{5}$ and $\cos B = \frac{4}{5}$.

(9) $\cot(A - B)$ when $\tan A = \frac{24}{7}$ and $\sin B = \frac{5}{13}$.

(10) $\sec(A - B)$ when $\sin A = \frac{60}{61}$ and $\cos B = \frac{12}{13}$.

Prove the following identities:—

(11) $\dfrac{\cos(A - B)}{\cos A \cos B} = 1 + \tan A \tan B.$

(12) $\tan(A - B) = \dfrac{\tan A \cot B - 1}{\cot B + \tan A}.$

(13) $\dfrac{\sin(A - B)}{\sin A \sin B} = \cot B - \cot A.$

(14) $\cos(A - B) \sec A \operatorname{cosec} B = \tan A + \cot B.$

(15) $\cot(A - B) = \dfrac{1 + \cot A \cot B}{\cot B - \cot A}.$

(16) $\cot(A - 45^\circ) = \dfrac{\tan A + 1}{\tan A - 1}.$

(17) $\sin^2(45 + A) + \sin^2(45 - A) = 1.$

(18) $\sin(A + C)\sin B - \sin(B + C)\sin A = \sin C \sin(B - A).$

(19) $\dfrac{\sin(A - B)\cos C}{\cos(B - B)\cos A} = \dfrac{\tan A - \tan B}{1 + \tan B \tan C}.$

(20) $\cos(B - C)\sin A - \sin(A - B)\cos C = \cos(A - C)\sin B.$

(21) If $\tan A = 2\tan B$, show that $\sin(A + B) = 3\sin(A - B)$.

(22) If $\tan A = \dfrac{\sqrt{3}}{4 - \sqrt{3}}$ and $\tan B = \dfrac{\sqrt{3}}{4 + \sqrt{3}}$

show that $\tan(A - B) = \cdot 375$.

54. The methods of this Chapter may be extended to finding the **trigonometrical ratios of the sum of three angles.**

(1) $\sin(A+B+C) = \sin(A+B)\cos C + \cos(A+B)\sin C$, treating $A+B+C$ as $(A+B)+C$, *i.e.* regarding $A+B$ as one angle in the first expansion.

$$= (\sin A \cos B + \cos A \sin B)\cos C + (\cos A \cos B - \sin A \sin B)\sin C,$$

$$= \sin A \cos B \cos C + \sin B \cos C \cos A + \sin C \cos A \cos B - \sin A \sin B \sin C.$$

(2) $\cos(A+B+C) = \cos(A+B)\cos C - \sin(A+B)\sin C$,

$$= (\cos A \cos B - \sin A \sin B)\sin C - (\sin A \cos B + \cos A \sin B)\sin C,$$

$$= \cos A \cos B \cos C - \sin A \sin B \cos C - \sin B \sin C \cos A - \sin C \sin A \cos B.$$

(3) $$\tan(A+B+C) = \frac{\tan(A+B) + \tan C}{1 - \tan(A+B)\tan C},$$

$$= \frac{\dfrac{\tan A + \tan B}{1 - \tan A \tan B} + \tan C}{\dfrac{1 - \tan A + \tan B}{1 - \tan A \tan B} - \tan C};$$

$$= \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A}.$$

If some of the angles are subtracted the same methods will still hold. For instance

$$\sin(A - B - C) = \sin(A - \overline{B+C})$$
$$= \sin A \cos(B+C) - \cos A \sin(B+C),$$

$$= \sin A \cos B \cos C - \sin B \cos C \cos A - \sin C \cos A \cos B - \sin A \sin B \sin C.$$

EXAMPLES VIII (*c*).

Prove the following :—

(1) $\dfrac{\sin (A + B + C)}{\cos A \cos B \cos C} = \tan A + \tan B + \tan C - \tan A \tan B \tan C.$

(2) $\dfrac{\cos (A + B + C)}{\sin A \sin B \sin C} = \cot A \cot B \cot C - \cot A - \cot B - \cot C.$

Express in terms of the sines and cosines of the single angles—

(3) $\sin (A + B - C)$.

(4) $\cos (B + C - A)$.

(5) Express $\cot (A + B + C)$ in terms of $\cot A$, $\cot B$, $\cot C$.

(6) Express $\tan (A + B - C)$ in terms of $\tan A$, $\tan B$, $\tan C$.

(7) If $A + B + C = 90°$, prove that $\cot A + \cot B + \cot C = \cot A \cot B \cot C$.

(8) Also that $\tan A \tan B + \tan B \tan C + \tan C \tan A = 1$.

55. The formulæ in the foregoing chapter may be deduced from one another as in the following example :—

To deduce the expansions of the trigonometrical ratios of the difference of two angles, when the expansions for the trigonometrical ratios of the sum of two angles are given.

For instance, let it be given that

$$\cos (X + B) = \cos X \cos B - \sin X \sin B.$$

Let $X = 90° - A$, then $\sin X = \cos A$, $\cos X = \sin A$.

$$\therefore \cos (X + B) = \cos (90° - A + B) = \cos \{ 90° - (A - B) \} = \sin (A - B)$$

$$\therefore \sin (A - B) = \sin A \cos B - \cos A \sin B.$$

Similarly the expansion of $\cos (A - B)$ may be deduced from that of $\sin (B + X)$ by supposing $X = 90° - A$.

For the tangent we may proceed as follows :—

Let $$Y = (A - B). \quad \therefore A = B + Y;$$

$$\tan A = \tan (B + Y) = \frac{\tan A + \tan Y}{1 - \tan B \tan Y}.$$

$$\therefore \tan A - \tan A \tan B \tan Y = \tan B + \tan Y,$$

whence $$\tan Y = \frac{\tan A - \tan B}{1 + \tan A \tan B},$$

or, $$\tan (A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}.$$

In a similar way, when the formulæ for the difference of two angles are given, we may obtain the formulæ for their sum.

CHAPTER IX.

MULTIPLE AND SUB-MULTIPLE ANGLES.

56. To obtain the trigonometrical ratios of the double angle in terms of those of the single angle.

In the expansion of sin (A + B), putting B = A, we obtain

$$\sin (A + A) = \sin A \cos A + \cos A \sin A,$$

or, $$\mathbf{\sin 2 A = 2 \sin A \cos A} \quad \ldots\ldots \quad (1)$$

In the expansion of cos (A + B), putting B = A, we obtain

$$\cos (A + A) = \cos A . \cos A - \sin A . \sin A,$$

or, $$\cos 2 A = \cos^2 A - \sin^2 A$$

$$= \cos^2 A - (1 - \cos^2 A), \text{ or } = (1 - \sin^2 A) - \sin^2 A$$

$$\therefore \mathbf{\cos 2 A = 2 \cos^2 A - 1, \text{ or } = 1 - 2 \sin^2 A} \quad \ldots \quad (2)$$

In the expansion of tan (A + B), putting B = A,

$$\tan (A + A) = \frac{\tan A + \tan A}{1 - \tan A . \tan A}$$

$$\therefore \mathbf{\tan 2 A = \frac{2 \tan A}{1 - \tan^2 A}} \quad \ldots\ldots \quad (3)$$

57. To obtain the trigonometrical ratios of the single angle in terms of those of the double angle.

From (2) in the preceding article we obtain at once

$$\sin^2 A = \frac{1 - \cos 2 A}{2} \quad \ldots\ldots \quad (4)$$

$$\cos^2 A = \frac{1 + \cos 2 A}{2} \quad \ldots\ldots \quad (5)$$

and dividing (5) by (4), $\tan^2 A = \dfrac{1 - \cos 2 A}{1 + \cos 2 A} \quad \ldots \quad (6)$

But tan A is usually expressed in another way as follows:—

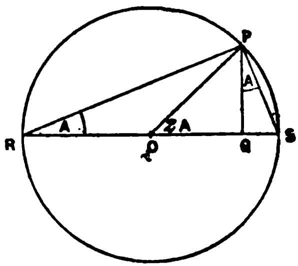
$$\tan A = \frac{\sin A}{\cos A} = \frac{2 \sin A \cos A}{2 \cos^2 A} = \frac{\sin 2A}{1 + \cos 2A} \quad . \quad . \quad (7)$$

a very useful form for finding the ratios of half a given angle. For instance, knowing sin 30° and cos 30°,

$$\tan 15^\circ = \frac{\sin 30^\circ}{1 + \cos 30^\circ} = \frac{1}{2 + \sqrt{3}} = 2 - \sqrt{3}$$

a result which has been found in another way in Examples VIII. (*b*).

58. On account of the special importance of the above formulæ we shall give geometrical proofs of them as well.



Take any circle with centre O, draw the diameter R O S. Make an angle P R S = A, join P O, P S, and draw P Q perpendicular to R S. Then the angle P O S at the centre = 2 A, then O P = O R = O S = $\frac{1}{2}$ R S, and $\angle$ Q P S = A. (Eucl. iii.)

(1) $\sin 2A = \frac{PQ}{OP} = \frac{2PQ}{RS}$

$$= 2\frac{PS}{RS} \cdot \frac{PQ}{PS} = 2 \sin A \cdot \cos A.$$

(2) $\cos 2A = \frac{OQ}{OP} = \frac{2OQ}{RS} = \frac{RQ - QS}{RS}$

$$= \frac{RQ \cdot PR}{PR \cdot RS} - \frac{QS \cdot PS}{PS \cdot RS}$$

$$= \cos A \cdot \cos A - \sin A \cdot \sin A$$

$$= \cos^2 A - \sin^2 A.$$

(3) $\tan 2A = \dfrac{PQ}{OQ} = \dfrac{2PQ}{RQ - QS} = \dfrac{\dfrac{2PQ}{RQ}}{1 - \dfrac{QS}{RQ}}$

$= \dfrac{\dfrac{2PQ}{RQ}}{1 - \dfrac{QS \cdot RQ}{RQ^2}}$, but $QS \cdot RQ = PQ^2$ (Eucl. vi. 8.)

$= \dfrac{\dfrac{2PQ}{RQ}}{1 - \dfrac{PQ^2}{RQ^2}} = \dfrac{2 \tan A}{1 - \tan^2 A}.$

(4) $\tan A = \dfrac{PQ}{RQ} = \dfrac{PQ}{RO + OQ} = \dfrac{PQ}{OP + OQ}$

$= \dfrac{\dfrac{PQ}{OP}}{1 + \dfrac{OQ}{OP}} = \dfrac{\sin 2A}{1 + \cos 2A}.$

59. *Example* 1.—To express $\sin 2A$ and $\cos 2A$ in terms of $\tan A$.

$\sin 2A = 2 \sin A \cos A = \dfrac{2 \sin A \cos A}{\cos^2 A + \sin^2 A}$, (since $\cos^2 A + \sin^2 A = 1$)

$= \dfrac{\dfrac{2 \sin A}{\cos A}}{1 + \dfrac{\sin^2 A}{\cos^2 A}}$, (dividing numerator and denominator by $\cos^2 A$)

$= \dfrac{2 \tan A}{1 + \tan^2 A}.$

Similarly $\cos 2A = \dfrac{\cos^2 A - \sin^2 A}{\cos^2 A + \sin^2 A} = \dfrac{1 - \dfrac{\sin^2 A}{\cos^2 A}}{1 + \dfrac{\sin^2 A}{\cos^2 A}}$

$= \dfrac{1 - \tan^2 A}{1 + \tan^2 A}.$

Example 2.—Given $\tan 2A = \frac{24}{7}$, find tan A,

$$\tan 2A = \frac{2\tan A}{1-\tan^2 A} = \frac{24}{7}.$$

$$\therefore \quad 12\tan^2 A + 7\tan A - 12 = 0,$$

$$(4\tan A - 3)(3\tan A + 4) = 0,$$

whence $$\tan A = \frac{3}{4} \text{ or } \left[-\frac{4}{3}\right].$$

Note.—The meaning of the negative value of tan A will be explained in § 78.

Example 3.—To show that $\tan(45° + A) + \tan(45° - A) = 2\sec 2A$.

$$\tan(45° + A) = \frac{\tan 45° + \tan A}{1 - \tan 45° \tan A} = \frac{1+\tan A}{1-\tan A},$$

similarly $$\tan(45° - A) = \frac{1-\tan A}{1+\tan A}.$$

$$\therefore \quad \tan(45° + A) + \tan(45° - A) = \frac{1+\tan A}{1-\tan A} + \frac{1+\tan A}{1-\tan A}$$

$$= \frac{2(1+\tan^2 A)}{1-\tan^2 A}$$

$$= \frac{2}{\cos 2A}.$$ (See *Example* 1.)

EXAMPLES IX (*a*).

Find the value of sin 2 A, cos 2 A, and tan 2 A in each of the following cases:—

(1) $\sin A = \frac{3}{5}$. (2) $\cos A = \frac{24}{25}$.

(3) $\tan A = \frac{9}{40}$. (4) $\cos A = \frac{\sqrt{5}}{3}$.

(5) $\tan A = \frac{1}{2}$. (6) $\sin A = \frac{11}{61}$.

Find sin A, cos A, and tan A when—

(7) $A = 22\frac{1}{2}°$. (8) $\cos 2A = \frac{7}{9}$.

(9) $\sin 2A = \frac{4}{5}$. (10) $\tan 2A = \frac{1 - m^2}{2m}$.

Prove the following identities:—

(11) $\sin 2A = \dfrac{2 \cot A}{1 + \cot^2 A}$.

(12) $\cos 2A = \dfrac{\cot^2 A - 1}{\cot^2 A + 1}$.

(13) $\cot A - \cot 2A = \operatorname{cosec} 2A$.

(14) $2 \cos A = \sin A \sin 2A + 2 \cos^3 A$.

(15) $\sin^2 2A = 2 \cos^2 A (1 - \cos 2A)$.

(16) $\sec^2 A (1 + \sec 2A) = 2 \sec 2A$.

(17) $\cot \dfrac{A}{2} = \dfrac{\sin A}{1 - \cos A}$.

(18) $\cos 2A (1 + \tan 2A \tan A) = 1$.

(19) $\dfrac{1 + \sin\theta - \cos\theta}{1 + \sin\theta + \cos\theta} = \tan \dfrac{\theta}{2}$.

(20) $\cos 2A = \dfrac{\cot^2 A - \tan^2 A}{2 + \tan^2 A + \cot^2 A}$.

(21) $\dfrac{1 + \tan 2A \tan A}{\tan A + \cot A} = \dfrac{1}{2} \tan 2A$.

(22) $\cot\theta = \dfrac{1 + \sec 2\theta}{\tan 2\theta}$.

(23) $\tan 2\theta + \sec 2\theta = \dfrac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}$.

(24) $2 \cos^2 (45 - A) = 1 + \sin 2A$.

(25) $\tan 2A = \tan 2A \cot^2 A - 2 \cot A$.

(26) $\cos (A + B) \cos (A - B) + \sin (A + B) \sin (A - B) = \dfrac{1 - \tan^2 B}{1 + \tan^2 B}$.

(27) $\tan^2 \dfrac{A}{2} + \cot^2 \dfrac{A}{2} - 4 \cot^2 A = 2$.

(28) $\sin 4A = 4 \cos A (\sin A + 2 \sin^3 A)$.

(29) $\cos 4A = 1 - 8 \sin^2 A \cos^2 A$.

(30) $\tan 4A = \dfrac{4 \tan A - 4 \tan^3 A}{1 - 6 \tan^2 A + \tan^4 A}$.

---

60. To express sin 3 A, cos 3 A, and tan 3 A in terms of sin A, cos A and tan A respectively,

$$\sin 3A = \sin (2A + A) = \sin 2A \cos A + \cos 2A \sin A,$$
$$= 2 \sin A \cos A \cos A + (1 - 2 \sin^2 A) \sin A,$$

by (1) and (2) in § 56 $= \sin A \{2 \cos^2 A + 1 - 2 \sin^2 A\}$,

$$= \sin A \{3 - 4 \sin^2 A\},$$
$$= 3 \sin A - 4 \sin^3 A.$$

$$\cos 3A = \cos (2A + A) = \cos 2A \cos A - \sin 2A \sin A,$$
$$= (2 \cos^2 A - 1) \cos A - 2 \sin A \cos A \sin A,$$
$$= \cos A (2 \cos^2 A - 1 - 2 \sin^2 A),$$
$$= \cos A (4 \cos^2 A - 3),$$
$$= 4 \cos^3 A - 3 \cos A,$$

$$\tan 3A = \frac{\tan 2A + \tan A}{1 - \tan 2A \tan A} = \frac{\dfrac{2 \tan A}{1 - \tan^2 A} + \tan A}{1 - \dfrac{2 \tan A}{1 - \tan^2 A} . \tan A}.$$

by (3) in § 56 $= \dfrac{3 \tan A - \tan^3 A}{1 - 3 \tan^2 A}$.

Example 1.—To prove that

$$\cos 3A - \sin 3A = (\cos A + \sin A)(1 - 2 \sin^2 A).$$

$$\cos 3A - \sin 3A = 4 \cos^3 A - 3 \cos A - (3 \sin A - 4 \sin^3 A).$$
$$= 4 (\cos^2 A + \sin^3 A) - 3 (\cos A + \sin A).$$
$$= (\cos A + \sin A) \{4 (\cos^2 A + \sin^2 A - \sin A \cos A) - 3\}$$
$$= (\cos A + \sin A) \{4 - 4 \sin A \cos A - 3\}$$

since $\cos^2 A + \sin^2 A = 1$.

$$= (\cos A + \sin A)(1 - 2 \sin 2A),$$

since $2 \sin A \cos A = \sin 2A$.

Example 2 shows that

$$\tan 3\,A - \tan 2\,A - \tan A = \tan A \tan 2\,A \tan 3\,A,$$

$$\tan 3\,A = \tan(2\,A + A) = \frac{\tan 2\,A + \tan A}{1 - \tan 2\,A \tan A}.$$

$$\therefore\ \tan 3\,A - \tan 3\,A \tan 2\,A \tan A = \tan 2\,A + \tan A;$$

or $\tan 3\,A - \tan 2\,A - \tan A = \tan 3\,A \tan 2\,A \tan A.$

EXAMPLES IX (*b*).

Find the value of

(1) $\cos 3\,A$ when $\cos A = \frac{7}{8}$.

(2) $\sin 3\,A$ when $\sin A = \frac{1}{3}$.

(3) $\tan 3\,A$ when $\tan A = \frac{1}{2}$.

(4) $\tan 3\,A$ when $\tan 2\,A = \frac{3}{4}$ (one value only).

(5) $\cos 3\,A$ when $\cos 2\,A = \frac{17}{32}$ " " "

Prove that,

(6) $\cos 3\,A = 2\cos A \cos 2\,A - \cos A$.

(7) $\sin 3\,A = \sin A + 2\sin A \cos 2\,A$.

(8) $\dfrac{\cos 3\,A + \sin 3\,A}{\cos A - \sin A} = 1 + 2\sin 2\,A$.

(9) $\cot 3\,A = \dfrac{\cot^3 A - 3\cot A}{3\cot^2 A - 1}$,

(10) $4\cos^3 A \sin 3\,A - 4\sin^3 A \cos 3\,A = 3\sin 4\,A$.

(11) $\dfrac{\operatorname{cosec} A \sec 2\,A}{\operatorname{cosec} 3\,A} - \sec 2\,A = 2$.

(12) $\sin 5\,A - 5\sin 3\,A = 16\sin^5 A - 10\sin A$.

61. To find the sines and cosines of angles of 18° and 72°.

Let $18° = x$. Then, since we know that

$$\sin 36° = \cos 54°, \text{ its complement},$$

or
$$\sin 2x = \cos 3x,$$

$$\therefore 2 \sin x \cos x = 4 \cos^3 x - 3 \cos x,$$

dividing by $\cos x$, we get

$$2 \sin x = 4 \cos^2 x - 3 = 1 - 4 \sin^2 x.$$

$$\therefore 4 \sin^2 x + 2 \sin x - 1 = 0.$$

By solving this equation as a quadratic,

$$\sin x = \frac{-1 \pm \sqrt{5}}{4}.$$

Now the sines of all angles between 0 and 90° vary in value between 0 and 1, *i.e.*, are positive;

so we reject the value $\dfrac{-\sqrt{5}-1}{4}$ which is negative.

$$\therefore \sin 18° = \frac{\sqrt{5}-1}{4}.$$

$$\therefore \cos 72° = \sin(90° - 72°) = \sin 18° = \frac{\sqrt{5}-1}{4}$$

$$\cos 18° = \sqrt{1 - \sin^2 18°} = \sqrt{1 - \frac{(\sqrt{5}-1)^2}{16}}$$

$$= \frac{\sqrt{10 + 2\sqrt{5}}}{4}$$

$$\sin 72° = \cos(90° - 72°) = \cos 18° = \frac{\sqrt{10+2\sqrt{5}}}{4}.$$

From these values the sines and cosines of 36° and 54° can be deduced.

62. We have now the means of calculating the sines and cosines of 3° and its multiples, for

$$\sin 3° = \sin(18° - 15°) = \sin 18° \cos 15° - \cos 18° \sin 15°,$$
$$\sin 6° = \sin(36° - 30°),$$
$$\sin 9° = \sin(45° - 36°),$$
$$\sin 12° = \sin(72° - 60°),$$

and any higher unknown multiple of 3° may be expressed as the sum of two known multiples.

The approximate calculation of sin 3° may be performed as follows :—

We find

$$\sqrt{6} = 2{\cdot}4495,\ \sqrt{5} = 2{\cdot}2361\ \ldots;\ \sqrt{2} = 1{\cdot}4142,$$

whence

$$\sin 18^\circ = \frac{\sqrt{5}-1}{4} = {\cdot}3090\ \ldots$$

$$\cos 18^\circ = {\cdot}9510\ \ldots$$

$$\sin 15^\circ = \frac{\sqrt{3}-1}{2\sqrt{2}} = \frac{\sqrt{6}-\sqrt{2}}{4} = {\cdot}2588\ \ldots$$

$$\sin 15^\circ = \frac{\sqrt{3}+1}{2\sqrt{2}} = \frac{\sqrt{6}+\sqrt{2}}{4} = {\cdot}9659\ \ldots$$

$$\therefore \sin 3^\circ = {\cdot}3090 \times {\cdot}9659 - {\cdot}2588 \times {\cdot}9510 = {\cdot}2984 - {\cdot}2461$$
$$= {\cdot}0523.$$

EXAMPLES IX (*c*).

Prove the following :—

(1) $\cos 36^\circ = \dfrac{\sqrt{5}+1}{4}.$

(2) $\sin 36^\circ = \dfrac{1}{4}\sqrt{10 - 2\sqrt{5}}.$

(3) $\tan 36^\circ = \sqrt{5 - 2\sqrt{5}}.$

(4) $\cot 18^\circ = \sqrt{5 + 2\sqrt{5}}.$

(5) $\sin 54^\circ \cos 72^\circ = \sin^2 30^\circ.$

(6) $\tan 36^\circ \tan 72^\circ = 4 \cos 36^\circ - 1.$

Calculate approximately—

(7) $\cos 6^\circ.$

(8) $\sin 78^\circ.$

(9) $\tan 9^\circ.$

(10) $\cos 24^\circ.$

CHAPTER X.

RESOLUTION OF THE PRODUCTS OF TRIGONOMETRICAL RATIOS.

63. To express the products of the sines and cosines of two angles in terms of the sines and cosines of the sum and difference of the angles.

Let A and B be two angles, A being the greater. We have

$$\sin A \cos B + \cos A \sin B = \sin (A + B) \quad . \quad . \quad . \quad . \quad (1)$$
$$\sin A \cos B - \cos A \sin B = \sin (A - B) \quad . \quad . \quad . \quad . \quad (2)$$
$$\cos A \cos B - \sin A \sin B = \cos (A + B) \quad . \quad . \quad . \quad . \quad (3)$$
$$\cos A \cos B + \sin A \sin B = \cos (A - B) \quad . \quad . \quad . \quad . \quad (4)$$

adding (1) and (2) we get

$$2 \sin A \cos B = \sin (A + B) + \sin (A - B) \quad . \quad . \quad . \quad (5)$$

subtracting (2) from (1)

$$2 \cos A \sin B = \sin (A + B) - \sin (A - B) \quad . \quad . \quad . \quad (6)$$

adding (3) and (4)

$$2 \cos A \cos B = \cos (A + B) + \cos (A - B) \quad . \quad . \quad . \quad (7)$$

subtracting (3) from (4)

$$2 \sin A \sin B = \cos (A - B) - \cos (A + B) \quad . \quad . \quad . \quad (8)$$

Example 1.—

$$2 \cos 60^\circ \sin 15^\circ = \sin (60^\circ + 15^\circ) - \sin (60^\circ - 15^\circ) \text{ by } (6)$$
$$= \sin 75^\circ - \sin 45^\circ.$$

Example 2.—Simplify cos 6 A cos A + sin 4 A sin A.

$$\cos 6A \cos A = \frac{1}{2}(\cos 7A + \cos 5A) \quad \ldots \quad \text{by (7)}$$

$$\sin 4A \sin A = \frac{1}{2}(\cos 3A - \cos 5A) \quad \ldots \quad \text{by (8)}$$

$$\therefore \cos 6A \cos A + \sin 4A \sin A$$
$$= \frac{1}{2}(\cos 7A + \cos 5A + \cos 3A - \cos 5A)$$
$$= \frac{1}{2}(\cos 7A + \cos 3A).$$

ample 3.—Show that

$$\frac{\tan A}{\tan B} = \frac{\sin(A+B) + \sin(A-B)}{\sin(A+B) - \sin(A-B)}.$$

$$\frac{\tan A}{\tan B} = \frac{\sin A \cdot \cos B}{\cos A \cdot \sin B} = \frac{\frac{1}{2}\{(\sin(A+B) + \sin(A-B))\}}{\frac{1}{2}\{(\sin(A+B) - \sin(A-B))\}}$$
$$= \frac{\sin(A+B) + \sin(A-B)}{\sin(A+B) - \sin(A-B)}.$$

EXAMPLES X.

Express the following products in terms of the sums or differences of the ratios :—

(1) $2 \cos 2A \cos A$.

(2) $2 \sin 3A \cos 5A$.

(3) $2 \sin \frac{7A}{2} \sin \frac{3A}{2}$.

(4) $2 \sin 3A \cos 2B$.

(5) $2 \sin 5A \cos 2A$.

(6) $\cos \frac{3A}{2} \cos \frac{A}{2}$.

(7) $\sin 15° \cos 45°$.

(8) $\sin 45° \cos 15°$.

(9) $2 \sin (x + 3y) \cos (x - 3y)$.

(10) $2 \cos\left(45° - \frac{A}{2}\right) \cos\left(45 + \frac{A}{2}\right)$.

Prove the following identities:—

(11) $\tan 3A \tan A = \frac{\cos 2A - \cos 4A}{\cos 2A + \cos 4A}$.

(12) $\tan (A + 30°) \tan (A - 30°) = \frac{1 - 2 \cos 2A}{1 + 2 \cos 2A}$.

(13) $\tan (45° - A) \tan (45° - 3A) = \frac{1 - 2 \sin 2A}{1 + 2 \sin 2A}$.

(14) $\frac{\tan\left(45° - \frac{A}{2}\right)}{\tan\left(45° + \frac{A}{2}\right)} = \frac{\sec A - \tan A}{\sec A + \tan A}$.

(15) $\sin (A + 2B) \sin A - \sin (B + 2A) \sin B = \cos^2 B - \cos^2 A$.

(16) $\cos^2 A - \sin (30° + A) \sin (30° - A) = \frac{3}{4}$.

CHAPTER XI.

ON COMPOUNDING THE SUM AND DIFFERENCE OF SINES AND COSINES.

64. From § 63 we have the following results:

$$\sin(A+B)+\sin(A-B) = 2\sin A\cos B,$$
$$\sin(A+B)-\sin(A-B) = 2\cos A\sin B,$$
$$\cos(A+B)+\cos(A-B) = 2\cos A\cos B,$$
$$\cos(A+B)-\cos(A+B) = 2\sin A\sin B.$$

If we put $\begin{array}{l} A+B=C \\ A-B=D \end{array}$ we have,

adding $\quad 2A = C+D$, or $A = \dfrac{C+D}{2}$;

subtracting $\quad 2B = C-D$, or $B = \dfrac{C-D}{2}$.

Substituting these values for $A+B$, $A-B$, A, and B, we have

$$\sin C+\sin D = 2\sin\frac{C+D}{2}\cos\frac{C-D}{2} \quad . \quad . \quad (9)$$

$$\sin C-\sin C = 2\cos\frac{C+D}{2}\sin\frac{C-D}{2} \quad . \quad . \quad (10)$$

$$\cos C+\cos D = 2\cos\frac{C+D}{2}\cos\frac{C-D}{2} \quad . \quad . \quad (11)$$

$$\cos D-\cos C = 2\sin\frac{C+D}{2}\sin\frac{C-D}{2} \quad . \quad . \quad (12)$$

By means of these four formulæ we can compound the sum or difference of the sines and cosines of two angles into products of the half-sum and half-difference of the angles.

Example 1.—Simplify

$$\frac{\sin 3\theta + \sin 5\theta}{\cos 3\theta + \cos 5\theta}$$

$$\sin 3\theta + \sin 5\theta = 2 \sin \frac{5\theta + 3\theta}{2} \cdot \cos \frac{5\theta - 3\theta}{2}$$

$$= 2 \sin 4\theta \cos \theta$$

similarly $\cos 3\theta + \cos 5\theta = 2 \cos 4\theta \cos \theta$

$$\therefore \frac{\sin 3\theta + \sin 5\theta}{\cos 3\theta + \cos 5\theta} = \frac{2 \sin 4\theta \cos \theta}{2 \cos 4\theta \cos \theta}$$

$$= \frac{\sin 4\theta}{\cos 4\theta} = \tan 4\theta.$$

Example 2.—Show that

$\sin \theta \cos 2\theta + \sin 2\theta \cos 5\theta = \cos 4\theta \sin 3\theta$.

$$\sin \theta \cos 2\theta = \frac{1}{2}\{\sin(2\theta + \theta) - \sin(2\theta - \theta)\} = \frac{1}{2}(\sin 3\theta - \sin \theta),$$

similarly

$$\sin 2\theta \cos 5\theta = \frac{1}{2}(\sin 7\theta - \sin 3\theta).$$

$$\therefore \sin \theta \cos 2\theta + \sin 2\theta \cos 5\theta = \frac{1}{2}(\sin 3\theta - \sin \theta + \sin 7\theta - \sin 3\theta)$$

$$= \frac{1}{2}(\sin 7\theta - \sin \theta),$$

$$= \cos 4\theta \sin 3\theta.$$

EXAMPLES XI (*a*).

Simplify the following expressions by applying the formulæ of § 64.

(1) $\dfrac{\sin 4A + \sin 2A}{\cos 4A + \cos 2A}$.

(2) $\dfrac{\sin A + \sin 3A}{\cos A - \cos 3A}$.

(3) $\dfrac{\sin 5A - \sin A}{\cos A + \cos 5A}$.

(4) $\dfrac{\cos 3A - \cos 7A}{\sin 7A + \sin 3A}$.

(5) $\dfrac{\sin 2A - \sin 4B}{\cos 4B - \cos 2A}$.

(6) $\dfrac{\sin\dfrac{3A}{2} + \sin\dfrac{7A}{2}}{\cos\dfrac{3A}{2} - \cos\dfrac{7A}{2}}$.

(7) $\dfrac{\sin\dfrac{5A}{4} - \sin\dfrac{A}{4}}{\cos\dfrac{5A}{4} + \cos\dfrac{A}{4}}$.

(8) $\dfrac{\cos 5A - \cos\dfrac{11A}{2}}{\sin 5A + \sin\dfrac{11A}{2}}$.

(9) $\dfrac{\cos 2A + 2\cos 4A + \cos 6A}{\cos 4A + 2\cos 6A + \cos 8A}$.

(10) $\dfrac{\cos A - \sin A - \cos 3A + \sin 3A}{\sin 2A + \cos 2A}$.

Prove the following:

(11) $\cos 10A + \cos 4A + 2\cos 3A \cos A = 4\cos 4A \cos^2 3A$.

(12) $\sin 3A + \sin 5A - 2\sin 4A \cos 5A$
$= 4\sin 4A \sin 3A \sin 2A$.

(13) $\dfrac{\sin A + 2\sin 3A + \sin 5A}{\cos A - 2\cos 3A + \cos 5A} = \dfrac{4\sin A - 3\operatorname{cosec} A}{4\cos A - 3\sec A}$.

(14) $\cos 3A + \sin 2A = 2\sin\left(45^\circ - \dfrac{A}{2}\right)\cos\left(45^\circ - \dfrac{5A}{2}\right)$.

(15) $\dfrac{\sin A + \sin 3A + \sin 5A + \sin 7A}{\cos A + \cos 3A + \cos 5A + \cos 7A} = \tan 4A$.

(16) $\dfrac{\cos(x - 3y) - \cos(3x - y)}{\sin 2x + \sin 2y} = 2\sin(x - y)$.

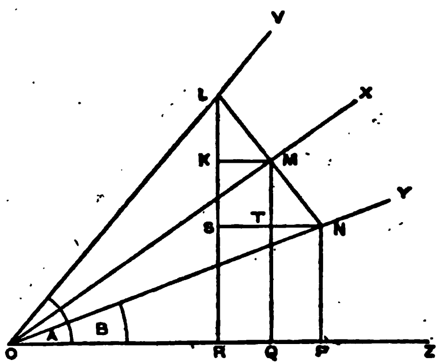
(17) $\dfrac{\sin(x + 3y) + \sin(3x + y)}{\sin 2x + \sin 2y} = 2\cos(x + y)$.

(18) $\cos 15^\circ - \sin 15^\circ = \dfrac{1}{\sqrt{2}}$.

(19) $\dfrac{\sin nA - \sin(n-2)A}{\cos(n-2)A - \cos nA} = \cot(n-1)A$.

(20) $\dfrac{l\sin(A-B) + m\sin A + l\sin(A+B)}{l\cos(A-B) + m\cos A + l\cos(A+B)} = \tan A$.

65. On account of the importance of the theorems in § 64 we give direct geometrical proofs as well.



(1) The sum of the sines of two angles equals twice the product of the sine of the half-sum of the angles, into the cosine of the half-difference.

Let $\angle VOZ = A$,

$\angle YOZ = B$,

then $\angle VOY = A - B$.

Bisect $\angle VOY$ by OX,

then $\angle XOV = \angle XOY = \dfrac{A-B}{2}$,

$$\angle XOZ = B + \frac{A-B}{2} = \frac{A+B}{2}$$

In OX take any point M and draw LMN perpendicular to OX, meeting OV in L, OY in N. From L, M, N draw LR, MQ, NP perpendicular to OZ, and from M and N draw MK, NS, perpendicular to LR.

The two triangles L O M, M O N are equal, having the side O M common, the angles L O M, M O N equal, and the angles at M right angles;

$$\therefore \quad LM = MN, \text{ and } OL = ON. \quad \text{Eucl. (i. 26.)}$$

also

$$LK = KS, \text{ and } NT = ST. \quad \text{Eucl. (vi. 2.)}$$

Then

$$\sin A + \sin B = \frac{LR}{OL} + \frac{NP}{ON},$$
$$= \frac{LR + NP}{OL} = \frac{\overline{MQ + LK} + \overline{MQ - MT}}{OL},$$
$$= \frac{2\,MQ}{OL}, \text{(since } LK = KS = MT)$$
$$= \frac{2\,MQ}{OM} \cdot \frac{OM}{OL},$$
$$= 2 \sin XOZ \,.\, \cos XOV,$$
$$= 2 \sin \frac{A+B}{2} \,.\, \cos \frac{A-B}{2}.$$

(2) The sum of the cosines of two angles equals twice the product of the cosine of the half-sum of the angles into the cosine of the half-difference.

Using the same figure and construction as before,

$$\cos A + \cos B = \frac{OR}{OL} + \frac{OP}{ON} = \frac{OR + OP}{OL},$$
$$= \frac{\overline{OQ - RQ} + \overline{OQ + QP}}{OL}$$
$$= \frac{2\,OQ}{OL}, \text{[since } PQ = QR \because NT = TS.]$$
$$= \frac{2\,OQ}{OM} \cdot \frac{OM}{OL},$$
$$= 2 \cos XOZ \,.\, \cos XOY,$$
$$= 2 \cos \frac{A+B}{2} \,.\, \cos \frac{A-B}{2}.$$

(3) The difference of the sines of two angles equals twice the product of the cosine of the half-sum of the angles into the sine of the half-difference.

Using the same figure and construction,

$$\sin A - \sin B = \frac{LR}{OL} - \frac{NP}{ON} = \frac{LR - NP}{OL} = \frac{LS}{OL},$$
$$= \frac{2\,LK}{OL},$$
$$= \frac{2\,LK}{LM} \cdot \frac{LM}{OL},$$
$$= 2 \cos KLM \,.\, \sin VOX.$$

now

$$\angle KLM = 90^\circ - LMK = KMO = MOZ = \frac{A+B}{2}.$$

$$\therefore \quad \sin A - \sin B = 2 \cos \frac{A+B}{2} \,.\, \sin \frac{A-B}{2}$$

(4) **The difference of the cosines of two angles equals twice the product of the sine of the half-sum of the angles into the sine of the half-difference.**

Use the same figure and construction as before, but notice that the cosine of the smaller angle B $\left(\text{viz. } \frac{OP}{ON}\right)$, is greater than that of the greater angle A $\left(\text{viz. } \frac{OR}{OL}\right)$, O N and O L being equal; in order, therefore, to obtain a positive result we must subtract cos A from cos B, but in the result the half-difference of the angles will be $\frac{A-B}{2}$, since A is $>$ B.

$$\begin{aligned}\cos B - \cos A &= \frac{OP}{ON} - \frac{OR}{OL} = \frac{RP}{OL} = \frac{2KM}{OL},\\ &= \frac{2KM \,.\, LM}{LM \,.\, OL},\\ &= 2 \sin KLM \,.\, \sin XOV,\\ &= 2 \sin \frac{A+B}{2} \,.\, \sin \frac{A-B}{2}.\end{aligned}$$

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66. Certain kinds of **equations** can be solved by applications of the principles of this and the two preceding chapters.

Example 1.—Solve the equation

$$\cos \theta - \cos 7\theta = \sin 3\theta.$$

By application of § 64 (12) we get

$$2 \sin 4\theta \sin 3\theta = \sin 3\theta.$$

$$\therefore \quad (2 \sin 4\theta - 1) \sin 3\theta = 0.$$

$$\therefore \quad \sin 3\theta = 0, \text{ whence } \theta = 0,$$

$$\text{or,} \quad \sin 4\theta = \frac{1}{2}, \quad \text{,,} \quad 4\theta = \frac{\pi}{6}.$$

$$\theta = \frac{\pi}{24}.$$

Example 2.—Solve the equation—

$$\sin 3\theta = 2 \sin^3 \theta;$$

using § 60, we get

$$3 \sin \theta - 4 \sin^3 \theta = 2 \sin^3 \theta.$$

$$\therefore \quad 6 \sin^3 \theta = 3 \sin \theta,$$

$$\text{or,} \quad (2 \sin^2 \theta - 1) \sin \theta = 0.$$

$$\therefore \quad \sin \theta = 0, \text{ whence } \theta = 0,$$

$$\text{or,} \quad \sin \theta = \frac{1}{\sqrt{2}}, \quad \text{,,} \quad \theta = \frac{\pi}{4}.$$

EXAMPLES XI (*b*).

Solve the following equations:—

(1) $\cos 4\theta + \cos 2\theta = \cos\theta$.

(2) $\cos 3x - \cos 2x + \cos x = 0$.

(3) $\cos x - \cos 9x = \sin 5x$.

(4) $\cos x + \cos 3x + \cos 5x + \cos 7x = 0$.

(5) $\cos nx + \cos (n-2)x = \cos x$.

(6) $\tan(a+x)\tan(a-x) = \dfrac{1-\cos 2a}{1+\cos 2a}$.

(7) $\sin(x+a) = 2\cos^2(x-a) - 1$.

(8) $\tan 3\theta - \tan\theta = \tan 4\theta - \tan 2\theta$.

(9) $\sin \overline{m+1}\,\theta - \sin m\theta = \cos n\theta - \cos \overline{n+1}\,\theta$.

(10) $\tan 2x + \tan 3x + \sqrt{3}\tan 2x \tan 3x = \sqrt{3}$.

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67. **Sin A + cos B can be expressed as a product of ratios** by writing

$$\cos B = \sin(90° - B).$$

thus $\sin A + \cos B = \sin A + \sin(90° - B)$

$$= 2\sin\left(45° + \frac{A}{2} - \frac{B}{2}\right)\cos\left(\frac{A}{2} + \frac{B}{2} - 45°\right).$$

Again, $1 + \sin A$ may be written

$$\sin 90° + \sin A = 2\sin(45° + A)\cos(45° - A).$$

Example 1.—Simplify $\dfrac{\sin 2A + \cos 2A}{\sin 2A - \cos 2A}$.

$$\sin 2A + \cos 2A = \sin 2A + \sin(90° - 2A)$$
$$= 2\sin 45° \,.\, \cos(2A - 45°).$$

Similarly

$$\sin 2A - \cos 2A = 2\cos 45^\circ \cdot \sin(2A - 45^\circ).$$

$$\therefore \quad \frac{\sin 2A + \cos 2A}{\sin 2A - \cos 2A} = \tan 45^\circ \cdot \frac{\cos(2A - 45^\circ)}{\sin(2A - 45^\circ)}$$

$$= \cot(2A - 45^\circ).$$

Example 2.—Prove by the principle of compounding that $1 + \cos 2A = 2\cos^2 A$.

$$1 + \cos 2A = \cos 0^\circ + \cos 2A = 2\cos\frac{2A + 0^\circ}{2} \cdot \cos\frac{2A - 0^\circ}{2}$$

$$= 2\cos^2 A.$$

EXAMPLES XI (*c*).

Express as products of trigonometrical ratios—

(1) $\sin 2A + \cos 2A$.

(2) $\cos A - \sin A$.

(3) $\cos(A + B) + \sin(A - B)$.

(4) $1 + \cos A$.

(5) $1 + \sin 3A$.

(6) $1 + 2\sin 2A$.

(7) $1 - 2\cos 2A$.

(8) $1 - \sqrt{2}\sin A$.

Prove the following identities :—

(9) $\cot 2A = \dfrac{2\tan(45^\circ - A)}{1 - \cot^2(45 + A)}$.

(10) $\dfrac{\tan A + 1}{\tan A - 1} = \dfrac{\sin(A + 45^\circ)}{\sin(A - 45^\circ)}$.

(11) $\tan\theta + \sec\theta = \tan\left(\dfrac{\pi}{4} + \dfrac{\theta}{2}\right)$.

(12) $\dfrac{1 + 2 \sin 2\theta}{1 - 2 \sin 2\theta} = \dfrac{\tan\left(\dfrac{\pi}{12} + \theta\right)}{\tan\left(\dfrac{\pi}{12} - \theta\right)}$.

(13) $\dfrac{\tan \dfrac{A}{2} + \cot \dfrac{A}{2} + 2}{\tan \dfrac{A}{2} + \cot \dfrac{A}{2} - 2} = \dfrac{\sin^2\left(\dfrac{A}{2} + 45^\circ\right)}{\sin^2\left(\dfrac{A}{2} - 45^\circ\right)}$.

(14) $\dfrac{2 \operatorname{cosec} 2\theta - \sec \theta}{2 \operatorname{cosec} 2\theta + \sec \theta} = \cot^2\left(\dfrac{\pi}{4} + \dfrac{\theta}{2}\right)$.

(15) $\dfrac{1 - \sqrt{3} \sin 2A}{\sin 2A} = \dfrac{\sin(30^\circ - A)}{\sin A} - \dfrac{\cos(30^\circ + A)}{\cos A}$.

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68. To show that

$$\sin(A + B)\sin(A - B) = \sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A.$$

$$\sin(A + B)\sin(A - B) = \frac{1}{2}\{\cos 2B - \cos 2A\}$$

$$= \frac{1}{2}\{1 - 2\sin^2 B - 1 + 2\sin^2 A\}$$

$$= \sin^2 A - \sin^2 B = 1 - \cos^2 A - 1 + \cos^2 B$$

$$= \cos^2 B - \cos^2 A.$$

EXAMPLES XI (*d*).

(1) Show that

$$\cos(A + B)\cos(A - B) = \cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A.$$

(2) Show that

$$\sin 2A \sin 2B = \sin^2(A + B) - \sin^2(A - B).$$

(3) Show that

$$\tan(A + 45^\circ)\tan(A - 45^\circ) = \frac{2\sin^2 A - 1}{2\cos^2 A - 1}.$$

(4) Solve the equations

$$\sin (A + B) \sin (A - B) = \frac{1}{2}$$

$$\cos (A + B) \cos (A - B) = 0.$$

(5) Simplify

$$\{\cos (A + B) + \sin (A + B)\} \{\cos (A - B) + \sin (A - B)\}.$$

(6) Show that

$$\frac{(\sin^2 A - \sin^2 B)^2}{\sin^2 (A + B)} + \frac{(\cos^2 A - \sin^2 B)^2}{\cos^2 (A + B)} = 1.$$

(7) Show that

$$(\sin^2 A - \sin^2 B) \cot (A + B) + (\sin^2 B - \sin^2 C) \cot (B + C) + (\sin^2 C - \sin^2 A) \cot (C + A) = 0.$$

69. We collect for purposes of reference the principal formulæ established in the three preceding chapters.

$$(\alpha)\begin{cases} \sin(A+B) = \sin A \cos B + \cos A \sin B. \\ \cos(A+B) = \cos A \cos B - \sin A \sin B. \\ \tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}. \\ \sin(A-B) = \sin A \cos B - \cos A \sin B. \\ \cos(A-B) = \cos A \cos B + \sin A \sin B. \\ \tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}. \end{cases}$$

$$(\beta)\begin{cases} \sin 2A = 2 \sin A \cos A. \\ \cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A. \\ \tan 2A = \dfrac{2 \tan A.}{1 - \tan^2 A}. \end{cases}$$

These all obtained from (α) by putting B = A.

$$(\gamma)\begin{cases} \sin 3A = 3 \sin A - 4 \sin^3 A. \\ \cos 3A = 4 \cos^3 A - 3 \cos A. \\ \tan 3A = \dfrac{3 \tan A - \tan^3 A}{1 - 3 \tan^2 A}. \end{cases}$$

By putting B = 2 A in (α) and substituting values given in (β).

$$(\delta)\begin{cases} 2 \sin A \cos B = \sin(A+B) + \sin(A-B). \\ 2 \cos A \sin B = \sin(A+B) - \sin(A-B). \\ 2 \cos A \cos B = \cos(A+B) + \cos(A-B). \\ 2 \sin A \sin B = \cos(A-B) - \cos(A+B). \end{cases}$$

From (α) by addition and subtraction.

$$(\epsilon)\begin{cases} \sin C + \sin D = 2 \sin \dfrac{C+D}{2} \cos \dfrac{C-D}{2}. \\ \sin C - \sin D = 2 \cos \dfrac{C+D}{2} \sin \dfrac{C-D}{2}. \\ \cos C + \cos D = 2 \cos \dfrac{C+D}{2} \cos \dfrac{C-D}{2}. \\ \cos D - \cos C = 2 \sin \dfrac{C+D}{2} \sin \dfrac{C-D}{2}. \end{cases}$$

From δ by writing C for (A + B) and D for (A − B.)

MISCELLANEOUS EXAMPLES ON PART II.

Prove the following identities :—

(1) $(\cot A - \tan A) \tan 2A = 2.$

(2) $\sin 2\theta + \sin 4\theta + \sin 6\theta = 4 \cos \theta \cos 2\theta \sin 3\theta.$

(3) $\tan 2A = \tan 2A \cot^2 A - 2 \cot A.$

(4) $\cot A + \tan 2A = \frac{1}{2} \tan 2A \operatorname{cosec}^2 A.$

(5) $\sin 35^\circ + \cos 65^\circ = \cos 5^\circ.$

(6) $\sin 70^\circ = \sin 10^\circ + \sin 50^\circ.$

(7) $\sin 4A = \tan 2A (1 + \cos 4A).$

(8) $\cos \theta + \cos 2\theta + \cos 3\theta + \cos 4\theta = 4 \cos \frac{\theta}{2} \cos \theta \cos \frac{5\theta}{2}.$

(9) $\dfrac{\sin A + \sin (A + B) + \sin (A + 2B)}{\cos A + \cos (A + B) + \cos (A + 2B)} = \tan (A + B).$

(10) $\cot \dfrac{A}{2} - \cot A = \dfrac{1}{\sin A}.$

(11) $\dfrac{1 + \cos 2A \cos 2B}{2} = \cos^2 A \cos^2 B + \sin^2 A \sin^2 B.$

(12) $\dfrac{1 + \sin 2A}{1 + \cos 2A} = \dfrac{1}{2} (1 + \tan A)^2.$

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(13) If $\tan A = \dfrac{3}{4}$ find the value of $3 \cos 2A + 4 \sin 2A.$

(14) If $\sin B + \sin C = a$, $\cos B + \cos C = b$,

show that $\tan \dfrac{B + C}{2} = \dfrac{a}{b}.$

(15) If $\sin 2\theta = b$, $\tan \theta = a$, then $2a = b(a^2 + 1).$

(16) Prove that $\sin^2 \theta$, $\frac{1}{2} \sin 2\theta$ and $\cos^2 \theta$ are in geometrical progression.

(17) If $\tan A = \frac{1}{3}$, $\tan B = \frac{1}{2}$, show that 45° is a value of $(A + B).$

(18) If $\cot \dfrac{A}{2} = 2 + \sqrt{3}$ find $\cos A.$

(19) Show that $\sin 2A$ can never be greater than $2 \sin A.$

(20) If $\tan A$, $\tan B$, $\cot A$, are in arithmetical progression, then $\sin 2A = \cot B$.

(21) If $\sin(A - B)$, $\sin B$, $\sin(A + B)$ are in G. P., then $\sin A = \sqrt{2} \sin B$.

(22) Solve the equation $1 - \cos 2\theta = \sqrt{2} \sin \theta$.

(23) If $\tan \alpha = \frac{5}{12}$, $\sin \frac{1}{2}\beta = \frac{3}{5}$, find $\sin(\alpha + \beta)$.

(24) If $\tan \theta = \frac{b}{a}$, prove that $a \cos 2\theta + b \sin 2\theta = a$.

(25) Eliminate θ from the equations $\cos \theta + \sin \theta = a$, $\sin 2\theta = b$.

(26) Show that $\cos 3A - \sin 3A$ is divisible by $\cos A + \sin A$.

(27) Simplify
$$\{\cos(A + B) + \sin(A + B)\}\{\cos(A - B) - \sin(A - B)\}.$$

(28) If α, β, γ, are in A. P., show that $\sin \alpha + \sin \gamma = 2 \sin \beta \cos(\beta - \alpha)$.

(29) If $\tan \theta = 2 \tan \phi$,
show that $\sin(\theta + \phi) = 3 \sin(\theta - \phi)$.

(30) If $\frac{a}{b} = \frac{\cos A}{\cos B}$, $a \tan A + b \tan B = (a + b) \tan \frac{A + B}{2}$.

(31) Solve the equation $\sin^2 \theta + \sin^2 2\theta = 1$.

Prove the following identities:—

(32) $\tan 60^\circ - \tan 40^\circ - \tan 20^\circ = \tan 20^\circ \tan 40^\circ \tan 60^\circ$.

(33) $\sin 10^\circ + \sin 20^\circ + \sin 40^\circ + \sin 50^\circ = \sin 70^\circ + \sin 80^\circ$.

(34) $\sin^2(45 + A) + \sin^2(45 - A) = 1$.

(35) $\tan 50^\circ + \tan 40^\circ = 2 \sec 10^\circ$.

(36) $(\cos A + \sin A)(\cos 3A + \sin 3A) = \cos 2A(1 + 2 \sin 2A)$.

(37) $4 \cos^6 \theta + 4 \sin^6 \theta = 1 + 3 \cos^2 2\theta$.

(38) $\cos 10A + \cos 4A + 2 \cos 3A \cos A = 4 \cos 4A \cos^2 3A$.

(39) $2 \cos 2A \cos A - 2 \sin 4A \sin A = 2 \cos 3A \cos 2A$.

(40) $\sin^2(A + B) + \sin^2(A - B) = 1 - \cos 2A \cos 2B$.

(41) $\sin 9A - \sin 6A + \sin 3A = \sin 6A(2 \cos 3A - 1)$.

(42) $\tan(45 + A) - \tan(45 - A) = 2 \tan 2A$.

(43) $\cos(\alpha + \beta)\cos(\alpha - \beta) + \sin(\alpha + \beta)\sin(\alpha - \beta) = \frac{1 - \tan^2 \beta}{1 + \tan^2 \beta}$.

(44) $\dfrac{\cos 9^\circ + \sin 9^\circ}{\cos 9^\circ - \sin 9^\circ} = \tan 54^\circ.$

(45) $\cos 6\,A = 16\,(\cos^6 A - \sin^6 A) - 15 \cos 2\,A.$

(46) $\dfrac{\tan 5\,A - \tan 3\,A}{1 + \tan 5\,A \tan 3\,A} = \dfrac{2 \tan A}{1 - \tan^2 A}.$

(47) $\dfrac{\cos a + \cos 3\,a}{\cos 2\,a} = \dfrac{\cos \frac{a}{2} + \cos \frac{3\,a}{2}}{\cos \frac{a}{2}}.$

(48) $\sec 2\,A - \frac{1}{2} \tan 2\,A \sin 2\,A = \dfrac{\cot^2 A + \tan^2 A}{\cot^2 A - \tan^2 A}.$

(49) $\dfrac{\cos 2\,A - \cos 4\,A}{\sin 4\,A - \sin 2\,A} + \dfrac{\cos A - \cos 3\,A}{\sin A - \sin 3\,A} = \dfrac{\sin A}{\cos 2\,A \cos 3\,A}.$

(50) $(\sin \theta - \cos \theta)^4 + (\sin \theta + \cos \theta)^4 = 3 - \cos 4\,\theta.$

(51) $\cot^2 a - \tan^2 a = 4 \operatorname{cosec} 2\,a \cot 2\,a.$

(52) $\tan\left(30^\circ + \dfrac{\theta}{2}\right) \tan\left(30^\circ - \dfrac{\theta}{2}\right) = \dfrac{2 - \sec \theta}{2 + \sec \theta}.$

(53) $8 \sin a \sin 2\,a \sin 3\,a \sin 4\,a = 1 - \cos 6\,a - \cos 8\,a + \cos 10\,a.$

(54) $\cos 2\,A + \cos 2\,B + \cos 2\,C + \cos 2\,(A + B + C) = 4 \cos (B + C) \cos (C + A) \cos (A + B).$

(55) $\cot (A + B) + \cot (A - B) = \dfrac{\sin 2\,A}{\cos^2 B - \cos^2 A}.$

(56) $\tan A + \cot \dfrac{A}{2} = \dfrac{\operatorname{cosec}^2 \frac{A}{2}}{\cot \frac{A}{2} - \tan \frac{A}{2}}.$

(57) $\dfrac{2 \sin \theta - \sin 2\,\theta}{2 \sin \theta + \sin 2\,\theta} = \tan^2 \dfrac{\theta}{2}.$

(58) $\sin (\theta - a) \sin (\theta + a) - \sin (\theta - \beta) \sin (\theta + \beta) = \sin^2 \beta - \sin^2 a.$

(59) $\dfrac{\sin A + 2 \sin 3\,A + \sin 5\,A}{\cos A - 2 \cos 3\,A + \cos 5\,A} = - \tan 3\,A \cot^2 A.$

(60) $4\,(\cos^8 A - \sin^8 A) = 4 \cos 2\,A - \sin 2\,A \sin 4\,A.$

(61) $\cos A + \cos B + \cos (A + B) + \cos (B + C) = 4 \cos \dfrac{A + B + C}{2} \cos \dfrac{C}{2} \cos \dfrac{A - B}{2}.$

(62) If $\cos(A - B) = \frac{3}{10}\operatorname{cosec}(A + B)$ and $\sin(A - B) = \frac{1}{10}\sec(A + B)$, find $\sin A$ and $\sin B$.

(63) Solve the equation $\tan\theta - 2\tan 2\theta + \tan 3\theta = 0$.

(64) If $\tan^2\theta = 1 + 2\tan^2\phi$, then $\cos 2\phi = 1 + 2\cos 2\theta$.

(65) Show that $\sin 27^\circ$ lies between $\frac{9}{20}$ and $\frac{1}{2}$.

(66) If $\alpha + \beta = \theta$ show that

$$\cos^2\alpha + \cos^2\beta - 2\cos\alpha\cos\beta\cos\theta = \sin^2\theta.$$

(67) Given $\tan A = \frac{4}{3}$ $\tan B = \frac{5}{12}$, find the value of $\sin(A + B) + \cos(A - B)$.

(68) If $\sin A = \tan B$,

$$\text{then } \sin(A - B)\cos B = \sin 2B\sin^2\frac{A}{2}.$$

(69) Prove that $2\cos 11^\circ 15' = \sqrt{2 + \sqrt{2 + \sqrt{2}}}$.

(70) If $\cos(\theta - \alpha)$, $\cos\theta$, $\cos(\theta + \alpha)$ are in harmonical progression, show that $\cos\theta = \sqrt{2}\cos\frac{\alpha}{2}$.

(71) If $\cos(A - C)\cos B = \cos(A - B + C)$ prove that $\tan A$, $\tan B$, $\tan C$ are in harmonical progression.

(72) Eliminate θ from the equations

$$4x = 3a\cos\theta + a\cos 3\theta,$$
$$4y = 3a\sin\theta - a\sin 3\theta.$$

(73) Solve the equation $\cos 2n\theta + \cos 2(n - 1)\theta = \cos\theta$.

(74) If $8\sin A\sin B\sin C = 1$, and $A + B + C = 90^\circ$, prove that $\cos 2A + \cos 2B + \cos 2C = \frac{3}{2}$.

Prove the following identities:—

(75) $\dfrac{\tan\theta + \sec\theta}{\cot\theta + \operatorname{cosec}\theta} = \tan\left(\dfrac{\pi}{4} + \dfrac{\theta}{2}\right)\tan\dfrac{\theta}{2}$.

(76) $\sin^4\theta + 4\cos^4\theta = (1 + \cos^2\theta - \sin 2\theta)(1 + \cos^2\theta + \sin 2\theta)$.

(77) $(3\sin A - \sin 3A)^2 + (3\cos A + \cos 3A)^2 = 4(4 - 3\sin^2 2A)$.

(78) $\tan^2\left(\dfrac{\pi}{4} + \theta\right)(1 - \sin 2\theta) = \tan^2\left(\dfrac{\pi}{4} - \theta\right)(1 + \sin 2\theta)$.

(79) $\sin 2\alpha + \sin 2\beta + \sin 2\gamma - \sin 2(\alpha + \beta + \gamma) = 4\sin(\alpha + \beta)\sin(\beta + \gamma)\sin(\gamma + \alpha)$.

(80) $\dfrac{\tan a}{\tan 3a - \tan a} = 2 \sin\left(\dfrac{\pi}{6} + a\right) \sin\left(\dfrac{\pi}{6} - a\right)$.

(81) $(1 - \sin\theta)(1 - \sin\phi) = \left\{\sin\dfrac{\theta + \phi}{2} - \cos\dfrac{\theta - \phi}{2}\right\}^2$.

(82) $\cos 6^\circ \cos 54^\circ - \sin 36^\circ \sin 24^\circ + \cos 66^\circ \cos 6^\circ = \frac{3}{4}$.

(83) $4 \sin\frac{1}{2}(A + B + C) \cos\frac{1}{2}(B + C - A) \cos\frac{1}{2}(A - B + C) \sin\frac{1}{2}(A + B - C) = \sin^2 A + \sin^2 B - \sin^2 C + 2 \sin A \sin B \cos C$.

(84) $(\tan 4A + \tan 2A)(1 - \tan^2 3A \tan^2 A) = 2 \tan 3A \sec^2 A$.

(85) $\cos 3A = 4 \cos A \sin(30 - A) \sin(30 + A)$.

(86) $\dfrac{\cos^2 a \cos\theta + \sin^2 a}{\cos^2 a \cos\theta - \sin^2 a} = \dfrac{1 - \cos 2a \tan^2\dfrac{\theta}{2}}{\cos 2a - \tan^2\dfrac{\theta}{2}}$.

(87) $\cos 4\theta = \dfrac{\cot^2\theta - 6 + \tan^2\theta}{\cot^2\theta + 2 + \tan^2\theta}$.

(88) $\dfrac{1}{2}\tan\dfrac{\theta}{2} + \dfrac{1}{4}\tan\dfrac{\theta}{4} = \dfrac{1}{4}\cot\dfrac{\theta}{4} - \cot\theta$.

(89) $\dfrac{2(\sin A + \sin B)}{\left\{\cos\dfrac{A + B}{2} + \cos\dfrac{A - B}{2}\right\}^2} =$
$\left\{\tan\dfrac{A}{2} + \tan\dfrac{B}{2}\right\}\left\{1 + \tan\dfrac{A}{2}\tan\dfrac{B}{2}\right\}$.

(90) $\left\{1 + \sqrt{2}\sin\left(\dfrac{\pi}{4} - \theta\right)\right\}^2 = 4\cos^2\dfrac{\theta}{2}(1 - \sin\theta)$.

If $A + B + C = 90^\circ$, prove the following identities:—

(91) $\sin 2A + \sin 2B + \sin 2C = 4 \cos A \cos B \cos C$.

(92) $1 - \sin^2 A - \sin^2 B - \sin^2 C = 2 \sin A \sin B \sin C$.

(93) $\cos^2 A + \cos^2 B - \cos^2 C = 2 \cos A \cos B \sin C$.

(94) $\cot A + \cot B + \cot C = \cot A \cot B \cot C$.

---

(95) Solve the equation $\sin 9\theta + \sin 5\theta + 2\sin^2\theta = 1$.

(96) If $\tan\left(45^\circ - \dfrac{A}{2}\right) = \tan^2 B$

then $\sqrt{\sec A + \tan A} + \sqrt{\sec A - \tan A} = 2 \operatorname{cosec} 2B$.

(97) If $\sin A$ be the geometric mean between $\sin B$ and $\cos B$, prove that $\cos 2A = 2 \sin (45° - B) \cos (45° + B)$.

(98) Find $\cos x$ from the equation

$$\tan \alpha \tan x = \tan^2 (\alpha + x) - \tan^2 (\alpha - x).$$

(99) $\cot A = (a^3 + a^2 + a)^{\frac{1}{2}}$, $\cot B = (a + a^{-1} + 1)^{\frac{1}{2}}$, $\tan C = (a^{-3} + a^{-2} + a^{-1})^{\frac{1}{2}}$, show that $A + B = C$.

(100) If $\cot \dfrac{\alpha + \beta}{2} = \dfrac{b \sin \alpha}{a - b \cos \alpha}$, prove that each of these is also equal to $\dfrac{b \sin \beta}{a - b \cos \beta}$.

PART III.

CHAPTER XII.

CONVENTIONS OF SIGNS.

70. Let A B C be three points A B C on a straight road. Let A C = a, A B = b. Then a man walking from A to B and back again to A, although he has walked a distance 2 b, has made no progress. If we assume then that from A to B should be called $+ b$, and from B to A $- b$, we have a means of measuring the progress, $+ b - b = 0$. Again, if the man had walked from A to C, and then returned to B, his progress would be $a - (a - b) = b$, which we see to be geometrically true.

71. Again, if a ball be thrown vertically upward to a height h, and returns to its place of projection, although it has traversed a distance 2 h, it is neither higher nor lower than before. If we assume, then, that the upward direction is positive, and the downward negative, we may say that the change of height is $+ h - h = 0$. If a stone be thrown 150 feet vertically upwards from a cliff 100 feet high, and fall on the beach, its path may be thus described—it has first travelled + 150 feet, and then − 250 feet. $\therefore$ its final position is $150 - 250 = - 100$, *i.e.*, 100 ft. below the point of projection.

72. These ideas are embodied in the following convention :

Let X O X′ be a horizontal line, Y O Y′ a vertical meeting the first in O. (See Fig. 1, p. 92.)

Then the horizontal distance of any point to the right of Y O Y′ is counted a positive length, that of any point to the left of Y O Y′ a negative one.

Again, the vertical distance or height of any point above X O X′ is counted positive, the vertical distance of any point below X O X′ is assumed to be negative.

73. A similar convention is adopted with respect to angles. In Trigonometry an angle is always measured by the angular distance traversed by a line O Q revolving from its initial position O X. As was noticed in § 3 this method of measuring

Fig. 1.

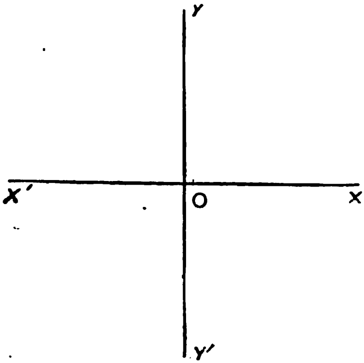
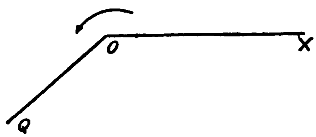
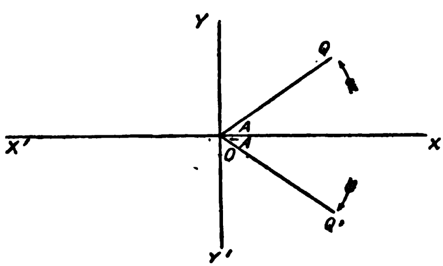


Fig. 2.



angles can be extended to angles of any size; for instance, when we speak of an angle of 945° we mean that the line O Q has revolved twice completely from O X (720°) and then through 225° more. (See Fig. 2.)

Further, if O Q revolve from O X in the opposite direction to the hands of a watch, the angle traced out is considered positive; if in the same direction as the hands of a watch the angle is considered negative. Thus, if in the figure O Q, O Q′ revolve in the directions indicated by the arrows through angular distances A, the angle Q O X is called A, and Q′ O X is called − A.



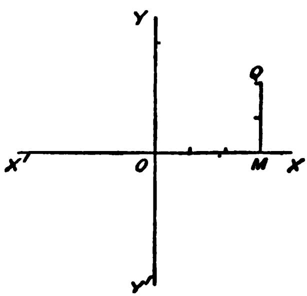
74. Each of the four spaces into which the plane of the paper is divided by the straight lines X O X′, Y O Y′ (con-

sidered of unlimited length) is called a quadrant; X O Y is called the first quadrant, Y O X′ the second, X′ O Y′ the third, and Y′ O X the fourth.

EXAMPLES XII.

(1) If $OA = a$, $OB = b$, $AC = x$, O A B C write down the algebraical values for the distances O C, C B, B A, (O B − B C), (O C − C B), (A C + C B).

(2) If Q M be the perpendicular from Q on X O X′ as in



this figure where $QM = 2$, $OM = 3$, draw diagrams showing the position of Q in the following cases, taking any unit of length you choose.

$$MQ = -2, \quad OM = +3$$
$$MQ = -2, \quad OM = -3$$
$$MQ = +2, \quad OM = -3$$
$$MQ = -3, \quad OM = -1$$
$$MQ = +4, \quad OM = +2$$
$$MQ = +5, \quad OM = -4$$
$$MQ = -3, \quad OM = +2.$$

(3) In what quadrant will O Q be in each case after revolving from O X through the following angles

$$125^\circ, \; -60^\circ, \; -212^g, \; \frac{5\pi}{4}, \; -\frac{5\pi}{3}, \; 250^g, \; 972^\circ, \; -\frac{12\pi}{5},$$

$$-1100^\circ, \; \frac{19\pi}{4}, \; 1310^g, \; \frac{130\pi}{7}.$$

(4) Draw diagrams showing the following angles as exactly as you can—

$$-120^\circ, \; 150^g, \; -\frac{5\pi}{4}, \; \frac{7\pi}{6}, \; 1200^\circ, \; -750^g, \; -\frac{20\pi}{3}, \; 3735^\circ.$$

(5) A stone is whirled round uniformly at the end of a string so as to make 45 revolutions per minute. Draw a figure indicating its position at the end of—

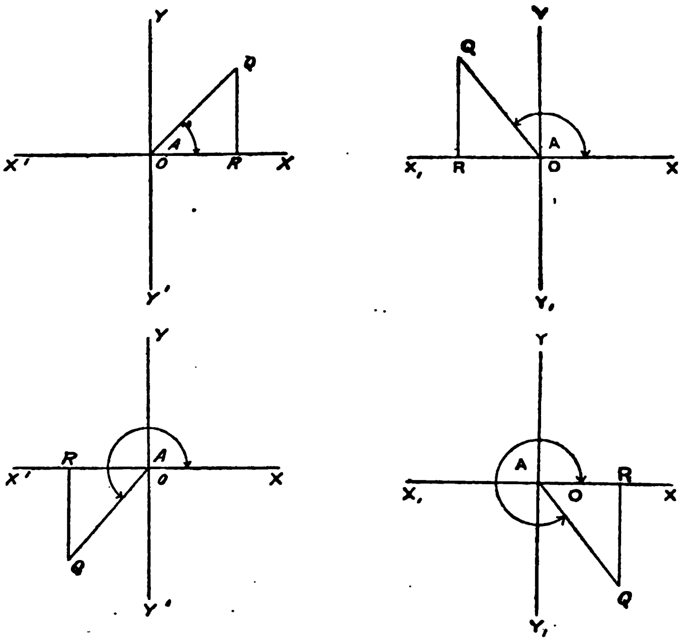
(1) the 45th second.
(2) 2 min. 20 secs.

CHAPTER XIII.

TRIGONOMETRICAL RATIOS OF ANGLES OF ANY SIZE.*

75. General definition of Trigonometrical Ratios.

Let the straight line O Q, revolving from the initial line O X take up its position in any one of the four quadrants. From Q



draw a perpendicular Q R on X O X′, then we obtain in each case a right-angled triangle Q O R, and the trigonometrical ratios of the angle Q O X (A) are formed by the ratios of the sides O R, R Q, O Q, of the triangle Q O R.

* Extension of Chapter III.

Thus in every quadrant—

$$\sin A = \frac{RQ}{OQ}, \qquad \operatorname{cosec} A = \frac{OQ}{RQ},$$

$$\cos A = \frac{OR}{OQ}, \qquad \sec A = \frac{OQ}{OR},$$

$$\tan A = \frac{RQ}{OR}; \qquad \cot A = \frac{OR}{RQ}.$$

76. In each case $RQ^2 + OR^2 = OQ^2$ (Euc. i. 47): and so, dividing out by OQ^2

$$\frac{RQ^2}{OQ^2} + \frac{OR^2}{OQ^2} = 1,$$

or $$\cos^2 A + \sin^2 A = 1;$$

similarly by dividing out by RQ^2 and OR^2 successively we get the formulæ

$$\operatorname{cosec}^2 A = 1 + \cot^2 A,$$

$$\sec^2 A = 1 + \tan^2 A,$$

which formulæ we established similarly in Chap. III. for angles in the first quadrant only.

77. In applying the convention of signs we only use it for lines parallel to XOX' and YOY'.

The revolving line O Q is always considered a positive length.

78. Remembering this we see

(1) That RQ is positive when Q lies in the first and second quadrants, negative when it lies in the third and fourth;

$\therefore \frac{RQ}{OQ}$, or the sine of any angle we may be considering, is positive in the first and second, negative in the third and fourth quadrants.

(2) OR is positive when Q lies in the first and fourth quadrants, negative when it lies in the second and third.

$\therefore \frac{OR}{OQ}$, or the cosine of any angle we may be considering, is positive in the first and fourth quadrants, negative in the second and third.

(3) $\frac{RQ}{OQ}$, or the tangent of the angle under consideration, will be positive when R Q and O Q have the same sign, *i.e.*, in the first and third quadrants; negative when they have unlike signs, in the second and fourth quadrants.

The following table sums up these results:

Y

2nd Quadrant.		1st Quadrant.	
sine	+	sine	+
cosine	−	cosine	+
tangent	−	tangent	+

X′ —— O —— X

3rd Quadrant.		4th Quadrant.	
sine	−	sine	−
cosine	−	cosine	+
tangent	+	tangent	−

Y′

EXAMPLES XIII.

What is the sign of the value of each of the following trigonometrical ratios?

(1) tan 105°.

(2) cos 295°.

(3) sin 150°.

(4) cos 240°.

(5) cosec (− 35°).

(6) tan $\frac{5\pi}{3}$.

(7) $\sec\left(-\frac{3\pi}{4}\right)$.

(8) $\cot\left(-\frac{12\pi}{5}\right)$.

(9) $\cos 840°$.

(10) $\sin(-560°)$.

(11) $\sec\frac{13\pi}{3}$.

(12) $\cos\left(-\frac{13\pi}{3}\right)$.

(13) $\tan\left(-\frac{27\pi}{8}\right)$.

(14) $\tan\frac{27\pi}{8}$.

(15) $\sin 1000°$.

(16) $\sin(-1000°)$.

(17) $\cot\left(2n\pi+\frac{2\pi}{3}\right)$ (n being a positive integer).

(18) $\sec\left(\overline{2n+1}\,\pi-\frac{\pi}{3}\right)$.

(19) $\sin\left(2n\pi.-\frac{2\pi}{5}\right)$.

(20) $\tan\left(n\pi-\frac{\pi}{3}\right)$.

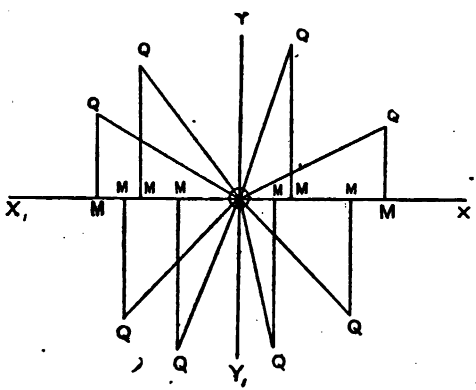
CHAPTER XIV.

VARIATION OF THE VALUES OF THE TRIGONOMETRICAL RATIOS.

79. Variation in the value of the sine as the angle changes.

By reference to the adjoining figure we see that as O Q revolves, the length of Q M varies from 0 to O Q, and having regard to signs that

from	0° to 90°	sin Q O M	changes	gradually from	0 to 1,
,,	90° to 180°	,,	,,	,,	1 to 0,
,,	180° to 270°	,,	,,	,,	0 to − 1,
,,	270° to 360°	,,	,,	,,	− 1 to 0.



80. Variation in the value of the cosine.

From the same figure we see that as O Q revolves the length of O M varies from O Q to 0; and that

from 0° to 90° cos Q O M changes gradually from 1 to 0,
" 90° to 180° " " " 0 to − 1,
" 180° to 270° " " " − 1 to 0,
" 270° to 360° " " " 0 to 1.

81. Variation of the value of the tangent.

Q M increases as O M diminishes, and *vice versâ;* in the initial position and when O Q has revolved through 180°, Q M = 0, O M = O Q: when O Q has revolved through 90° or 270°, Q M = O Q, O M = 0. Therefore, from

0° to 90° tan Q O M $\left(=\frac{QM}{OM}\right)$ changes gradually from 0 to ∞,

[see § 34]

90° to 180° " " " " − ∞ to 0,
180° to 270° " " " " 0 to ∞,
270° to 360° " " " " − ∞ to 0.

Note.—It ought to be noticed that Trigonometrical Ratios change from + to − or from − to + as they pass through the values 0 or ∞.

82. The variation of the other ratios can be found in a similar way, and the results may be tabulated in the following form :—

Angle.	0° to 90°.	90 to 180°.	180° to 270°.	270° to 360°.
sin . .	0 to 1	1 to 0	0 to −1	−1 to 0
cos . .	1 to 0	0 to −1	−1 to 0	0 to 1
tan . .	0 to ∞	− ∞ to 0	0 to ∞	− ∞ to 0
cot . .	∞ to 0	0 to − ∞	∞ to 0	0 to − ∞
sec . .	1 to ∞	− ∞ to −1	−1 to − ∞	∞ to 1
cosec .	∞ to 1	1 to ∞	− ∞ to −1	−1 to − ∞

83. We are now able to trace the change in value of trigonometrical expressions as the angle involved varies through all possible values.

Example 1.—Trace the change in value of $\cos 3\theta$ as θ varies from 0 to 180°.

when $\theta = 0°$	30°	60°	90°	120°	150°	180°
$3\theta = 0°$	90°	180°	270°	360°	450°	540°
$\cos 3\theta = 1$	0	-1	0	1	0	-1.

Example 2.—Trace the change in value of $\tan\theta + \cot\theta$ as θ changes from 0° to 180°.

We begin by expressing $\tan\theta + \cot\theta$ in terms of a single ratio, thus—

$$\tan\theta + \cot\theta = \frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta} = \frac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta} = \frac{2}{\sin 2\theta}$$

when	$\theta = 0°$	45°	90°	135°	180°
	$2\theta = 0°$	90°	180°	270°	360°
	$2\,\text{cosec}\,2\theta = \infty$	2	∞	2	∞

N.B.—When the value is ∞ it changes from + to −, or *vice versâ*; thus it increases from -2 to $-\infty$, then decreases from $+\infty$ to 2.

Example 3.—Trace the changes in value of $\cos x + \sin x$ between any limits of x.

Write $\cos x + \sin x = \cos x + \cos\left(\frac{\pi}{2} - x\right) = 2\cos\frac{\pi}{4}\cos\left(x - \frac{\pi}{4}\right)$

$= \sqrt{2}\cos\left(x - \frac{\pi}{4}\right)$, and then proceed as in Examples 1 and 2.

EXAMPLES XIV.

Trace the changes in the value of the following expressions:

(1) $\sin 2\theta$ as θ varies from 0° to 180°.

(2) $\tan\frac{\theta}{3}$ as θ varies from 90° to 180°.

(3) $\cos\left(\frac{\pi}{4} + 2\theta\right)$ as θ varies from 0° to 180°.

(4) $\sin\left(\theta - \frac{\pi}{4}\right)$ as θ varies from $90°$ to $450°$

(5) $\sec\frac{\theta}{2}$ as θ varies from $0°$ to $360°$.

(6) $\operatorname{cosec}\left(2\theta + \frac{\pi}{2}\right)$ as θ varies from $180°$ to $360°$.

(7) $\cos x + \sin x$ as x varies from $0°$ to $180°$.

(8) $\cos x - \sin x$ as x varies from $45°$ to $225°$.

(9) $\dfrac{\sin x}{1 + \cos x}$ as x varies from $0°$ to $360°$.

(10) $\sin x + \cos\left(\frac{\pi}{6} - x\right)$ as x varies from $\frac{\pi}{6}$ to $\frac{2\pi}{3}$.

(11) $\sin(\pi \sin x)$ as x increases from $0°$ to 2π.

(12) $\sin(\theta + \alpha) - \sin(\theta - \alpha)$ as θ increases from $90°$ to $270°$.

CHAPTER XV.

CONNECTION BETWEEN THE TRIGONOMETRICAL RATIOS OF VARIOUS ANGLES.

84. Since after any number of complete revolutions the line tracing out an angle returns to its original position, it is obvious that the trigonometrical ratios of any angle A are the same as those of

$$A + 360°,\ A + 720°, \text{ etc.}$$

and of

$$A - 360°,\ A - 720°, \text{ etc.}$$

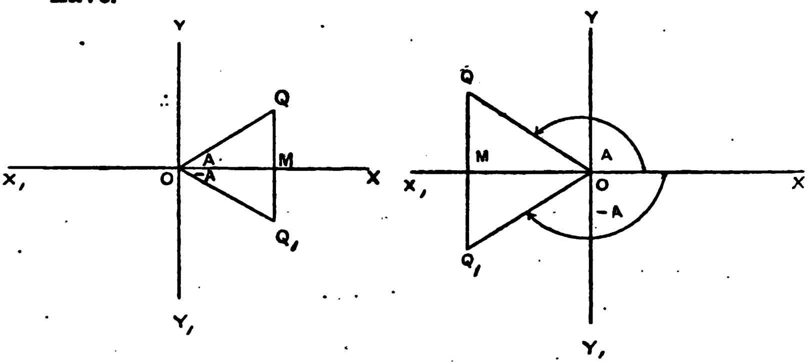
In general the ratios of A are the same as those of ($n\,360° + A$), or, as it is more often written, $2\,n\,\pi + \theta$, where n is any positive or negative integer, and θ is A expressed in circular measure.

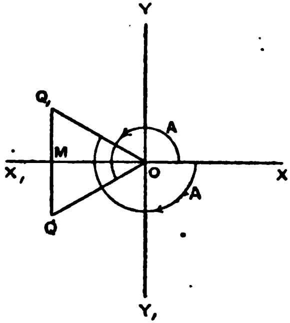
Example.

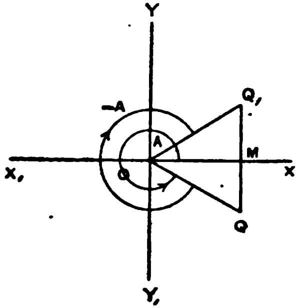
$$\sin 1500° = \sin(4 \times 360° + 60°) = \sin 60° = \frac{\sqrt{3}}{2}$$

$$\tan\frac{25\,\pi}{4} = \tan\left(6\,\pi + \frac{\pi}{4}\right) = \tan\frac{\pi}{4} = 1.$$

85. To find the trigonometrical ratios of an angle − A in terms of those of A, whatever magnitude A may have.







Let O Q revolving from its initial position O X trace out any positive angle. Let O Q_1 revolving from O X trace out the same angle in a negative direction. Then we shall have four figures according as O Q is in the 1st, 2nd, 3rd, or 4th quadrants. Take O Q = O Q_1, and draw perpendiculars from Q and Q_1 on X O X′. Then the triangles Q O M, Q_1 O M will be equal to one another in all respects, but M Q and M Q_1 will be always opposite in sign: so we have in each case

$$\left.\begin{aligned}\sin(-A) &= \sin Q_1OX = \frac{MQ_1}{OQ_1} = -\frac{MQ}{OQ} = -\sin A\\ \cos(-A) &= \cos Q_1OX = \frac{OM}{OQ_1} = \frac{OM}{OQ} = \cos A\\ \tan(-A) &= \tan Q_1OX = \frac{MQ_1}{OM} = -\frac{MQ}{OM} = -\tan A\end{aligned}\right\}(\alpha)$$

86. O Q_1 in the above figures also represents the position taken up by a line revolving from O X through 360° − A; therefore the trigonometrical ratios of Q_1 O X are those of 360° − θ. Whence we obtain at once from (α):—

$$\left.\begin{aligned}\sin(360^\circ - A) &= -\sin A\\ \cos(360^\circ - A) &= \cos A\\ \tan(360^\circ - A) &= -\tan A\end{aligned}\right\}(\beta)$$

87. To find the trigonometrical ratios of 180° − A in terms of those of A, whatever magnitude A may have.

Definition.— 180° − A is called the supplement of A.

Let a line O Q revolving from the initial position O X in either direction trace out an angle A. Let O Q_1 revolve through

180° in positive direction and then through an angle $-$ A. Then we shall have four figures according as O Q is in the 1st, 2nd, 3rd, or 4th quadrants.

Fig. 1.

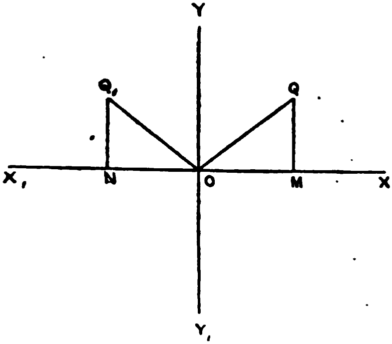


Fig. 2.

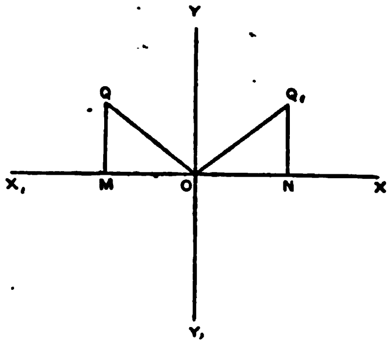


Fig. 3.

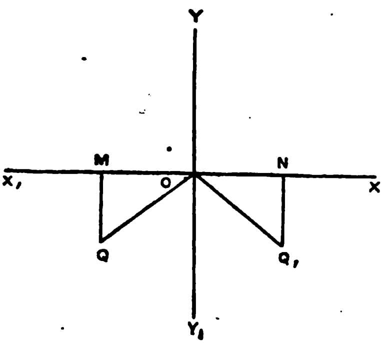
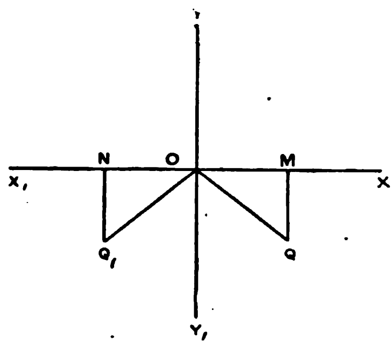


Fig. 4.



Take O Q_1 = O Q and draw perpendiculars Q M, Q_1 N on X O X_1. Then Q M O, Q_1 N O are equal right-angled triangles, and we have in each case

$$\left.\begin{aligned}\sin(180^\circ - A) &= \frac{NQ_1}{OQ_1} = \frac{MQ}{OQ} = \sin A\\ \cos(180^\circ - A) &= \frac{ON}{OQ_1} = \frac{-OM}{OQ} = -\cos A\\ \tan(180^\circ - A) &= \frac{NQ_1}{ON} = \frac{MQ}{-OM} = -\tan A\end{aligned}\right\}(\gamma)$$

Example.—
$$\sin 150° = \sin(180° - 30°) = \sin 30° = \frac{1}{2},$$
$$\cos 150° = \cos(180° - 30°) = -\cos 30° = -\frac{\sqrt{3}}{2},$$
$$\tan 150° = \tan(180° - 30°) = -\tan 30° = -\frac{1}{\sqrt{3}}.$$

88. Putting A = – B in the foregoing results we see that

$$\left.\begin{aligned} \sin(180° + B) &= \sin(-B) = -\sin B \\ \cos(180° + B) &= -\cos(-B) = -\cos B \\ \tan(180° + B) &= -\tan(-B) = \tan B \end{aligned}\right\} \quad (\delta)$$

Formulæ which we might have obtained independently in a manner similar to that of § 86.

Example.—
$$\sin 225° = -\sin 45° = -\frac{1}{\sqrt{2}},$$
$$\cos 240° = -\cos 60° = -\frac{1}{2},$$
$$\tan 225° = \tan 45° = 1.$$

EXAMPLES XV (*a*).

Find the values of

(1) $\sin 270°$.

(2) $\tan 750°$.

(3) $\sec \frac{2\pi}{3}$.

(4) $\cot 315°$.

(5) $\sin \frac{17\pi}{3}$.

(6) $\cos -\frac{5\pi}{3}$.

(7) $\operatorname{cosec}(-135°)$.

(8) $\tan\left(-\frac{7\pi}{3}\right)$.

(9) $\cot\left(-\frac{31\pi}{6}\right)$.

(10) versin 690°.

(11) $\sin 405° - \cos 585° + \sec 1215°$.

(12) $\dfrac{\tan 780° + \tan 495°}{1 + \cot(-390°)\cot 225°}$.

(13) Prove geometrically the results of § 88.

(14) Simplify $\sin(360° - \theta) + \sin(180° - \theta) + \sin(180° + \theta)$.

(15) Prove that $\sin(\beta - \gamma) + \sin(\gamma - \alpha) + \sin(\alpha - \beta) = -4\sin\frac{\alpha - \beta}{2}\sin\frac{\beta - \gamma}{2}\sin\frac{\gamma - \alpha}{2}$.

(16) Prove that $\cos(\beta - \gamma) + \cos(\gamma - \alpha) + \cos(\alpha - \beta) = 4\cos\frac{\beta - \gamma}{2}\cos\frac{\gamma - \alpha}{2}\cos\frac{\alpha - \beta}{2} - 1$.

(17) Prove that $\tan(\alpha - \beta) + \tan(\beta - \gamma) + \tan(\gamma - \alpha) = \tan(\alpha - \beta)\tan(\beta - \gamma)\tan(\gamma - \alpha)$.

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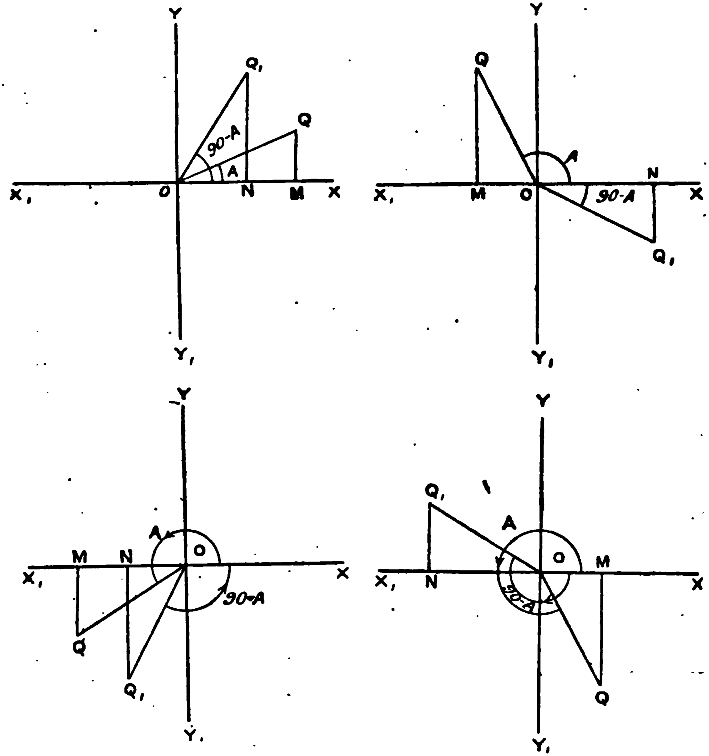
89. To prove that **sin A = cos (90° − A)**, &c., for all values of A.*



Let O Q revolving from its initial position trace out any angle A, positive or negative: then let O Q_1 trace out first an

* This theorem has been proved for the special case when A is less than 90° in § 29.

angle of 90° in a positive direction, then an angle A in the negative direction.

Then make $OQ_1 = OQ$ and draw QM, Q_1N perpendicular to XOX_1. Then in whatever quadrant OQ falls, OQ_1 will be in the same or the opposite quadrant, and QOM, Q_1ON being equal triangles (since the angle QOM is always the complement of the angle Q_1ON) we shall have $MQ = ON$, $NQ_1 = OM$ both in sign and magnitude.

Therefore

$$\sin A = \frac{MQ}{OQ} = \frac{ON}{OQ_1} = \cos(90° - A),$$

$$\cos A = \frac{OM}{OQ} = \frac{NQ_1}{OQ_1} = \sin(90° - A),$$

$$\tan A = \frac{MQ}{OM} = \frac{OM_1}{M_1Q_1} = \cot(90° - A),$$

and similarly for the other ratios.

90. We can in like manner prove the relation between the trigonometrical ratios for $(90° + A)$, $(270° + A)$, $(270° - A)$, and those for A, or we may deduce them thus:—

$\sin(90°+A) = \cos(-A)$ (by preceding theorem) $= \cos A$ by § 85.
$\cos(90°+A) = \sin(-A)$,, ,, $= -\sin A$,,
$\tan(90°+A) = \cot(-A)$,, ,, $= -\cot A$,,

$\sin(270°+A) = -\sin(90°+A)$ by § 88 $= -\cos A$ by this section.
$\cos(270°+A) = -\cos(90°+A)$,, $= \sin A$,,
$\tan(270°+A) = \tan(90°+A)$,, $= -\cot A$,,

$\sin(270°-A) = -\sin(90°-A)$,, $= -\cos A$ by § 89.
$\cos(270°-A) = -\cos(90°-A)$,, $= -\sin A$,,
$\tan(270°-A) = \tan(90°-A)$,, $= \cot A$,,

But the geometrical proofs, as independent, are preferable.

Example.—$\sin 105° = \sin(90° + 15°) = \cos 15° = \dfrac{\sqrt{3}+1}{2\sqrt{2}}$,

$$\cos 240° = \cos(270° - 30°) = \sin(-30°) = -\frac{1}{2}.$$

EXAMPLES XV (*b*).

(1) Prove geometrically that $\cos(90° + A) = -\sin A$ for all values of A.

(2) Also that $\sin(270° + A) = -\cos A$.

(3) Also that $\tan(270° - A) = \cot A$.

Simplify the following:—

(4) $\cos(270° + A) + \sin(180° + A) + \cos(90° + A)$.

(5) $\tan(270° - A) - \tan(90° + A) + \tan(270° + A)$.

(6) $\sin(90° + A) + \cos(180° + A) + \sin(270° + A)$.

(7) $\dfrac{\cos(270° - A)}{\sin(270° + A)} + \cot(270° - A) - \dfrac{\sin(180° + A)}{\cos(180° - A)}$.

(8) $\sin 300° + \cos 240° + \tan 225°$.

(9) $\sec\dfrac{2\pi}{3} - \operatorname{cosec}\dfrac{5\pi}{3} + \tan\dfrac{4\pi}{3}$.

Prove the following identities—

(10) $\sin(135° + A)\sin(135° - A) = \frac{1}{2}\cos 2A$.

(11) $\sin(270° - A)\sin(180° + A) = \frac{1}{2}\sin 2A$.

(12) $\cot\left(\dfrac{3\pi - \theta}{2}\right) = \dfrac{\cos\left(\dfrac{3\pi}{2} + \theta\right)}{\sin\dfrac{\pi}{2} + \sin\left(\dfrac{\pi}{2} + \theta\right)}$.

(13) $\dfrac{\sin 441° + \sin 171°}{\cos 351° - \cos 279°} = \cot 216°$.

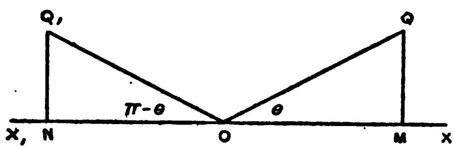
(14) $\tan^2\left(\dfrac{3\pi + \theta}{2}\right) - \cot^2\left(\dfrac{3\pi - \theta}{2}\right) = 4\sec\dfrac{\pi + 2\theta}{2}\cot(\pi - \theta)$.

(15) $\dfrac{2\cos\left(\dfrac{3\pi}{2} + \theta\right) - \cos\left(\dfrac{\pi}{2} - 2\theta\right)}{2\cos\left(\dfrac{\pi}{2} - \theta\right) + \cos\left(\dfrac{3\pi}{2} + 2\theta\right)} = \cot^2\left(\dfrac{3\pi + \theta}{2}\right)$.

CHAPTER XVI.

GENERAL VALUE FOR ANGLES WITH GIVEN TRIGONOMETRICAL RATIOS.

91. To find an expression for all the angles which have the same sine as a given angle θ.



Let Q O X be the given angle θ. Then the only other position, O Q_1, which will make $\frac{N\,Q_1}{O\,Q_1}$ **the same in magnitude and sign** as $\frac{M\,Q}{O\,Q}$, will be when O Q_1 lies in the second quadrant and $Q_1\,O\,X_1 = \theta$, that is, $Q_1\,O\,X = \pi - \theta$.

The first of these positions, O Q, will be attained after the revolving line has traced out angles

$$\left.\theta,\quad 2\pi+\theta,\quad 4\pi+\theta,\quad 6\pi+\theta, \text{ \&c., in a positive direction}\right\}(1)$$

or,

$$\left.-2\pi+\theta,\ -4\pi+\theta,\ -6\pi+\theta, \text{ \&c., in a negative direction}\right\}(2)$$

The other position, O Q_1 will be attained after revolution through angles

$$\left.\pi-\theta,\quad 3\pi-\theta,\quad 5\pi-\theta,\quad 7\pi-\theta, \text{ \&c., in a positive direction}\right\}(3)$$

$$\text{or, } \left.-\pi-\theta,\ -3\pi-\theta,\ -5\pi-\theta,\ -7\pi-\theta, \text{ \&c., in a negative direction}\right\}(4)$$

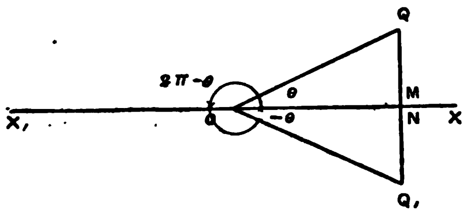
All these angles, and no others, are obtained by giving to n all possible integral values (including zero) in the formula

$$n\pi + (-1)^n \theta;$$

for when n is even, $(-1)^n$ is positive, and we get $+\theta$, as in (1) and (2);
when n is odd, $(-1)^n$ is negative, and we get $-\theta$, as in (3) and (4).

Therefore $n\pi + (-1)^n \theta$ is said to be the general formula for all angles which have the same sine as θ.

92. To find an expression for all the angles which have the same cosine as a given angle θ.



Let QOX be the given angle θ. Then the only other position, OQ_1, for which $\dfrac{ON}{OQ_1}$ will be **the same in sign and magnitude as** $\dfrac{OM}{OQ}$, will be when OQ_1 lies in the fourth quadrant and $Q_1OX = QOX$, that is, when the line has revolved through an angle $2\pi - \theta$.

The position OQ will be attained after the revolving line has traced out angles

$\theta,\quad 2\pi+\theta,\quad 4\pi+\theta,\quad 6\pi+\theta$, &c., in a positive direction . . (1)

or, $\quad -2\pi+\theta,\quad -4\pi+\theta,\quad -6\pi+\theta$, &c., in a negative direction . . (2)

The position OQ_1 will be attained after revolution through angles

$2\pi-\theta,\quad 4\pi-\theta,\quad 6\pi-\theta$, &c., in a positive direction . . (3)

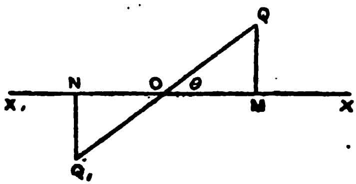
or, $\quad -\theta,\quad -2\pi-\theta,\quad -4\pi-\theta,\quad -6\pi-\theta$, &c., in a negative direction . . (4)

All these angles, and no others, are obtained by giving to n all possible integral values (including zero) in the formula

$$2\,n\,\pi \pm \theta.$$

Therefore $2\,n\,\pi \pm \theta$ is said to be the general formula for all angles which have the same cosine as θ.

93. To find an expression for all the angles which have the same tangent as θ.



Let QOX be the given angle θ. Then the only other position OQ_1 for which $\frac{NQ_1}{OM_1}$ will be the **same in sign and magnitude** as $\frac{MQ}{OM}$ is when OQ_1 lies in the third quadrant and $Q_1OX_1 = QOX$.

The position OQ will be attained after the revolving line has traced out angles

$\theta,\quad 2\pi+\theta,\quad 4\pi+\theta,\quad 6\pi+\theta$, &c., in a positive direction . . (1)

$-2\pi+\theta,\ -4\pi+\theta,\ -6\pi+\theta$, &c., in a negative direction . . (2)

The position OQ_1 will be attained after revolution through angles

$\pi+\theta,\quad 3\pi+\theta,\quad 5\pi+\theta,\quad 7\pi+\theta$, &c., in a positive direction . . (3)

$-\pi+\theta,\ -3\pi+\theta,\ -5\pi+\theta,\ -7\pi+\theta$, &c., in a negative direction . . (4)

All these angles, and no others, are included in the formula

$$n\pi+\theta$$

where n is zero or any integer, positive or negative.

$\therefore\ n\pi+\theta$ is said to be the general formula for all angles which have the same tangent as θ.

EXAMPLES XVI (*a*).

Give the general value of θ in the following cases:—

(1) $\cos\theta = \frac{1}{2}$.

(2) $\sin\theta = \frac{\sqrt{3}}{2}$.

(3) $\tan\theta = -1$.

(4) $\sec\theta = \sqrt{2}$.

(5) $\cos 2\theta = 0$.

(6) $\cot 3\theta = 1$.

(7) $\sin 2\theta = -\frac{1}{2}$.

(8) $\operatorname{cosec} 4\theta = \frac{2}{\sqrt{3}}$.

(9) $\sin\left(\frac{\pi}{4} + \theta\right) = \frac{1}{2}$.

(10) $\cos\left(\theta - \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$.

(11) $\tan\left(\frac{\pi}{3} + \theta\right) = 0$.

(12) $\cot\left(\theta - \frac{\pi}{6}\right) = \infty$.

(13) $\sin 2\theta = \cos\frac{\pi}{6}$.

(14) $\cos(\theta - a) = 1$.

(15) $\tan 3\theta = \cot a$.

(16) $\sin(\theta + a) = \cos(\theta - a)$.

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94. We now see that **the number of solutions of any trigonometrical equation is unlimited.**

Let us take the equation—

$$\tan^2\theta - (\sqrt{3} + 1)\tan\theta + \sqrt{3} = 0.$$

Solving in the ordinary way we find $\tan\theta = \sqrt{3}$ or 1.

Now $\frac{\pi}{3}$ and $\frac{\pi}{4}$ are the angles in the first quadrant whose tangents have these values.

If, however, we wish to give a general solution of this equation, *i.e.*, to include all the angles whose tangents are $\sqrt{3}$ and 1, we must write $\theta = n\pi + \frac{\pi}{3}$, or $n\pi + \frac{\pi}{4}$, where n denotes any integer from zero onwards. We see, therefore, that there is an unlimited number of solutions, the successive values of which, in this case, form two series in arithmetical progression whose common difference is π.

Example 1.—Solve the equation—

$$\cos 2\theta - \sqrt{3}\cos\theta + 1 = 0,$$

$$1 + \cos 2\theta = 2\cos^2\theta$$

$$\therefore\quad 2\cos^2\theta - \sqrt{3}\cos\theta = 0.$$

$$\therefore\quad \cos\theta = 0.$$

$$\text{or, } \cos\theta = \frac{\sqrt{3}}{2}.$$

The lowest values of θ are $\frac{\pi}{2}$ and $\frac{\pi}{6}$.

$\therefore$ the general solution is

$$\theta = 2n\pi \pm \frac{\pi}{2}, \text{ or } 2m\pi \pm \frac{\pi}{6}.$$

Example 2.—Solve

$$4\sqrt{2}(1 - \cos\theta) + \sqrt{3}\sec^2\frac{\theta}{2} = 2(\sqrt{6} + 2)\tan\frac{\theta}{2}.$$

This may be written

$$4\sqrt{2}(1 - \cos\theta) + \frac{2\sqrt{3}}{1 + \cos\theta} = 2(\sqrt{6} + 2)\frac{\sin\theta}{1 + \cos\theta},$$

whence, multiplying up,

$$4\sqrt{2}(1 - \cos^2\theta) + 2\sqrt{3} = 2(\sqrt{6} + 2)\sin\theta,$$

or,

$$4\sqrt{2}\sin^2\theta - 2(\sqrt{6} + 2)\sin\theta + 2\sqrt{3} = 0;$$

whence, solving in the ordinary way, we find

$$\sin\theta = \frac{\sqrt{3}}{2} \text{ or } \frac{1}{\sqrt{2}}.$$

$\therefore$ The lowest values of θ are $\frac{\pi}{6}$ or $\frac{\pi}{4}$, and the general solutions

$$\theta = n\pi + (-1)^n\frac{\pi}{6}.$$

$$\theta = n\pi + (-1)^n\frac{\pi}{4}$$

EXAMPLES XVI (*b*).

Solve the following equations, giving the general values in each case.

(1) $2 \sin \theta + 2 \operatorname{cosec} \theta + 5 = 0.$

(2) $\cos 5\theta + \cos 3\theta + \cos \theta = 0.$

(3) $1 + \sqrt{3} \tan^2 \theta = (1 + \sqrt{3}) \tan \theta.$

(4) $\cos \theta + \sqrt{3} \sin \theta = 2.$

(5) $2 \sin \theta + \sin 2\theta = 2 (1 + \cos \theta)^2.$

(6) $\sin \theta = \sqrt{2} \sin \dfrac{3\theta}{2}.$

(7) $\sqrt{\dfrac{1 + \sin^2 \theta}{2}} + \sqrt{\dfrac{1 + \cos^2 \theta}{2}} = \tan \dfrac{\pi}{3}.$

(8) $1 + \cos \theta + \cos 2\theta + \cos 3\theta = 0.$

(9) $(\cos \theta + \cos 3\theta)(\tan \theta + \tan 2\theta) = 2 \sin 5\theta.$

(10) $\tan (3x + a) + \tan (x + 3a) = 2 \tan 2 (x + a).$

(11) $\sqrt{2} \sin (x - 30^\circ) + \sqrt{2} \cos (x - 30^\circ) = \sqrt{3}.$

(12) $\tan \theta + \tan 4\theta + \tan 7\theta = \tan \theta \tan 4\theta \tan 7\theta.$

Construct the angles (θ) for which the three following equations are true.

(13) $12 \tan \theta - \cot \theta = 1.$

(14) $\tan 3\theta + \tan \theta = 0.$

(15) $\sec^4 \theta + 5 \tan^2 \theta + 11 = 0.$

(16) Show that the values of θ, derived from the equation $\sin p\theta + \sin q\theta = 0$, constitute two arithmetical progressions, the common differences of which are $\dfrac{2\pi}{p+q}$ and $\dfrac{2\pi}{p-q}$.

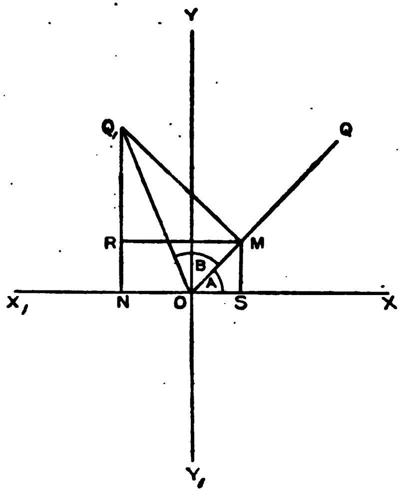
CHAPTER XVII.

EXTENSION OF THE FORMULÆ OF CHAPTER VIII. TO ANGLES OF ANY MAGNITUDE.

95. Sin (A + B) = sin A cos B + cos A sin B, when A + B is less than 180°, but greater than 90°.

Let $QOX = A$; $Q_1OQ = B$. From any point Q_1 in OQ_1 draw Q_1M, Q_1N perpendicular to OQ, OX respectively.

Through M draw MR, MS perpendicular to Q_1N and OX respectively.



The angle $RQ_1M = \frac{\pi}{2} - Q_1MR = RMO = MOX = A$.

$$\text{Then } \sin(A+B) = \frac{NQ_1}{OQ_1} = \frac{NR + RQ_1}{OQ_1} = \frac{MS}{OQ_1} + \frac{RQ_1}{OQ_1}.$$

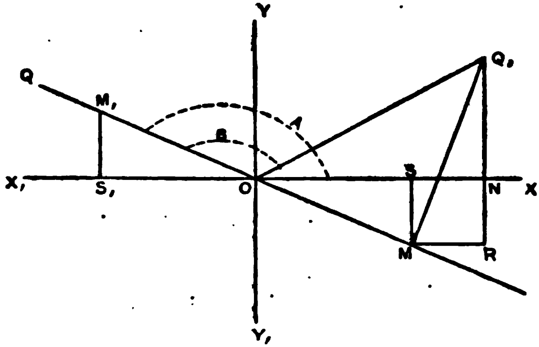
$$= \frac{MS}{OM} \cdot \frac{OM}{OQ_1} + \frac{RQ_1}{OM} \cdot \frac{OM}{OQ_1}$$

$$= \sin A \cos B + \cos A \sin B.$$

96. Cos (A − B) = cos A cos B + sin A sin B, when A and B are each less than 180°.

Let $QOX = A$, $QOQ_1 = B$, then $Q_1OX = A - B$. From Q_1 any point in OQ_1 draw perpendiculars Q_1M, Q_1N on QO produced and on OX respectively.

From M draw MS perpendicular to OX, MR perpendicular to Q_1N produced.



On OX_1 take $OS_1 = SO$ and draw M_1S_1 perpendicular to OX_1 meeting OQ in M_1.

Then $\cos(A - B) = \cos Q_1OX = \dfrac{ON}{OQ_1} = \dfrac{OS + SN}{OQ_1}$,

$$= \frac{OS}{OQ_1} + \frac{MR}{OQ_1} = \frac{OS \cdot OM}{OM \cdot OQ_1} + \frac{MR \cdot MQ_1}{MQ_1 \cdot OQ_1}.$$

Now $\dfrac{OS}{OM} = -\dfrac{OS_1}{OM_1} = -\cos A.$

$$\frac{OM}{OQ_1} = \cos Q_1OM = \cos(180° - B) = -\cos B.$$

$$\frac{MR}{MQ_1} = \sin MQ_1R = \sin Q_1ON = \sin(180° - A) = \sin A.$$

$$\frac{MQ_1}{OQ_1} = \sin MOQ_1 = \sin(180° - B) = \sin B.$$

$$\therefore \cos(A - B) = (-\cos A) \times (-\cos B) + \sin A \sin B$$
$$= \cos A \cos B + \sin A \sin B.$$

97. We are now able to expand $\sin(A \pm B)$, $\cos(A \pm B)$, $\tan(A \pm B)$ where A and B are angles of any magnitude. For A may be written $m \cdot 180° \pm C$, B may be written $n\,180° \pm D$, where C and D are each less than $90°$. $\therefore C + D$ less than $180°$.

The reader will understand the methods by the following examples:—

Example 1.—Expand $\sin(A + B)$ when A lies between $180°$ and $270°$ and B between $270°$ and $360°$.

Here $A = 180° + C$, $B = 360° - D$, where C and D are each less than a right angle.

$$\therefore \sin A = -\sin C \qquad \sin B = -\sin D.$$

$$\cos A = -\cos C \qquad \cos B = \cos D.$$

$$\begin{aligned}\text{Sin}(A + B) &= \sin(540° + C - D) = \sin(180° + C - D)\\ &= -\sin(C - D) = -\sin C\cos D + \cos C\sin D\\ &= \sin A\cos B + \cos A\sin B.\end{aligned}$$

Example 2.—Expand $\tan(\theta + \phi)$ when $\theta = n\pi + \alpha$, $\phi = m\pi - \beta$, α and β being each less than a right angle.

$$\begin{aligned}\tan(\theta + \phi) &= \tan\{(m + n)\pi + \alpha - \beta\} = \tan(\alpha - \beta)\\ &= \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}.\end{aligned}$$

Now $\tan\theta = \tan(n\pi + \alpha) = \tan\alpha$ by § 93.

$$\tan\phi = \tan(m\pi - \beta) = -\tan\beta.$$

$$\therefore \tan(\theta + \phi) = \frac{\tan\theta + \tan\phi}{1 - \tan\theta \cdot \tan\phi}.$$

Example 3.—To prove that

$$\sin X + \sin Y = 2\sin\frac{X + Y}{2}\cos\frac{X - Y}{2}$$

when X lies between $270°$ and $360°$, Y between $90°$ and $180°$.

Here $X = 360° - A$, $Y = 180° - B$, where A and B are each less than a right angle.

$$\begin{aligned}\sin X + \sin Y &= \sin(360° - A) + \sin(180° - B)\\ &= -\sin A + \sin B\\ &= 2\cos\frac{A + B}{2} \cdot \sin\frac{B - A}{2}\end{aligned}$$

now $(A + B) = 540° - (X + Y)$.

$$\therefore \cos\frac{A+B}{2} = \cos\left(270^\circ - \frac{X+Y}{2}\right) = -\sin\frac{X+Y}{2}, \text{ by § 90.}$$

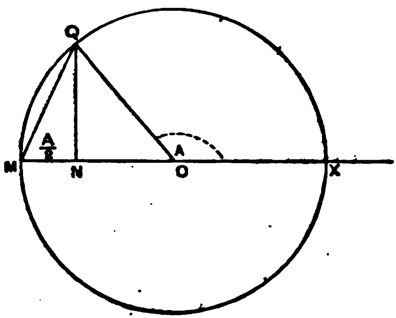
and, $B - A = X - Y - 180^\circ$.

$$\therefore \sin\frac{B-A}{2} = \sin\left(\frac{X-Y}{2} - 90^\circ\right) = -\cos\frac{X-Y}{2} \text{ by §§ 85, 89.}$$

$$\therefore \sin X + \sin Y = 2\sin\frac{X+Y}{2} \cdot \cos\frac{X-Y}{2}.$$

98. All the formulæ of Chapter VIII. can be thus extended. Any special case can of course be proved geometrically; for example:

Prove that $\tan\frac{A}{2} = \frac{\sin A}{1+\cos A}$ when A lies between 90° and 180°.



Let Q O X be equal to A; with O as centre, and with any radius describe a circle Q M X, cutting X O produced in M. Join Q M, then the angle Q M X $= \frac{A}{2}$. Draw Q N perpendicular to O M.

$$\text{Now} \quad \tan\frac{A}{2} = \frac{QN}{MN} = \frac{QN}{MO - NO}$$

$$= \frac{\frac{QN}{OQ}}{\frac{MO}{OQ} - \frac{NO}{OQ}} = \frac{\sin(180^\circ - A)}{1 - \cos(180^\circ - A)}$$

$$= \frac{\sin A}{1+\cos A}.$$

EXAMPLES XVII (*a*).

Prove geometrically the formulæ for the following expansions:—

(1) $\cos(A+B) = \cos A \cos B - \sin A \sin B$, when $A+B$ lies between 90° and 180°.

(2) $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$, under the same limitations.

(3) $\sin(A-B) = \sin A \cos B - \cos A \sin B$, when A and B are both less than 180°.

(4) $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$, under the same limitations.

(5) $\tan\dfrac{A}{2} = \dfrac{\sin A}{1+\cos A}$, when A lies between 180° and 270°.

(6) $\cot\dfrac{A}{2} = \dfrac{\sin A}{1-\cos A}$, when A lies between 90° and 180°.

Establish the formulæ for the expansion of—

(7) $\cos(A-B)$, when A lies between 270° and 360°, B between 180° and 270°.

(8) $\tan(A-B)$, when A lies between 90° and 180°, B between 180° and 270°.

(9) $\cos(A+B)$, when A lies between 180° and 270°, B between 360° and 450°.

(10) $\sin(A+B)$, when $A = m\pi - \alpha$, $B = n\pi + \beta$.

(11) $\sin(A-B)$, when $A = m\pi + \alpha$, $B = 2n\pi - \beta$.

Prove the following:—

(12) $\cos A + \cos B = 2\cos\dfrac{A+B}{2}\cos\dfrac{A-B}{2}$, when A lies between 90° and 180°, B between 270° and 360°.

(13) $\sin A - \sin B = 2\sin\dfrac{A-B}{2}\cos\dfrac{A+B}{2}$, when A lies between 180° and 270°, B between 360° and 450°.

(14) $\cos B - \cos A = 2 \sin \frac{A-B}{2} \sin \frac{A+B}{2}$, when A lies between 270° and 360°, B between 90° and 180°.

(15) $\sin(A+B) + \sin(A-B) = 2 \sin A \cos B$, when $A = 200°$, $B = 80°$.

Trigonometrical Properties connecting the Angles of a Triangle.

99. Many identities involving three angles A, B, C, can be obtained when A, B, C are the angles of a triangle; for they are then connected by the relation $A + B + C = 180°$, so that we may write $A + B = 180° - C$, or $\frac{A+B}{2} = \left(90 - \frac{C}{2}\right)$, &c.

$$\therefore \quad \sin\frac{A+B}{2} = \sin\left(90° - \frac{C}{2}\right) = \cos\frac{C}{2} \quad . \quad . \quad . \quad \S\ 89$$

$$\cos\frac{A+B}{2} = \cos\left(90° - \frac{C}{2}\right) = \sin\frac{C}{2} \quad . \quad . \quad . \quad ,,$$

$$\sin(A+B) = \sin(180° - C) = \sin C \quad . \quad . \quad . \quad \S\ 87$$

$$\cos(A+B) = \cos(180° - C) = -\cos C \quad . \quad . \quad ,,$$

The use of these modifications will be best understood by the following examples:—

Example 1.—$\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$ where A, B, C are the angles of a triangle.

$$\begin{aligned} &\sin 2A + \sin 2B + \sin 2C \\ &\qquad = 2\sin(A+B)\cos(A-B) + 2\sin C\cos C \\ &\qquad = 2\sin(180° - C)\cos(A-B) + 2\sin C\cos C \\ &\qquad = 2\sin C\,\{\cos(A-B) + \cos C\} \\ &\qquad = 2\sin C\,\{\cos(A-B) + \cos(180° - \overline{A+B})\} \\ &\qquad = 2\sin C\,\{\cos(A-B) - \cos(A+B)\} \\ &\qquad = 4\sin A\sin B\sin C. \end{aligned}$$

Example 2.—$\cos^2 A + \cos^2 B + \cos^2 C + 2\cos A\cos B\cos C = 1$, when $A + B + C = 180°$.

Here we have

$$\cos(A+B) = \cos(180° - C) = -\cos C,$$

$$\therefore \quad \cos A \cos B - \sin A \sin B = -\cos C,$$

or,

$$\cos A \cos B + \cos C = \sin A \sin B = \sqrt{1-\cos^2 A}\,\sqrt{1-\cos^2 B};$$

squaring, we get

$$\cos^2 A \cos^2 B + 2\cos A \cos B \cos C + \cos^2 C$$
$$= 1 - \cos^2 A - \cos^2 B + \cos^2 A \cos^2 B,$$

or, $\cos^2 A + \cos^2 B + \cos^2 C + 2\cos A \cos B \cos C = 1.$

Example 3.—Show that $\sin B + \sin C - \sin A = 4\cos\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$, when $A + B + C = 180°$.

$$\sin B + \sin C - \sin A$$
$$= 2\sin\frac{B+C}{2}\cos\frac{B-C}{2} - 2\sin\frac{A}{2}\cos\frac{A}{2}$$
$$= 2\sin\left(90° - \frac{A}{2}\right)\cos\frac{B-C}{2} - 2\sin\frac{A}{2}\cos\frac{A}{2}$$
$$= 2\cos\frac{A}{2}\left\{\cos\frac{B-C}{2} - \sin\frac{A}{2}\right\}$$
$$= 2\cos\frac{A}{2}\left\{\cos\frac{B-C}{2} - \sin\left(90 - \frac{B+C}{2}\right)\right\}$$
$$= 2\cos\frac{A}{2}\left\{\cos\frac{B-C}{2} - \cos\frac{B+C}{2}\right\}$$
$$= 4\cos\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$$

Example 4.—

$\tan A + \tan B + \tan C = \tan A \tan B \tan C$, when $A + B + C = 180°$.

We have

$$\tan(A+B) = \tan(180° - C) = -\tan C$$

$$\frac{\tan A + \tan B}{1 - \tan A \tan B} = -\tan C,$$

whence, multiplying up, we obtain the required result.

EXAMPLES XVII (*b*).

Prove the following identities, when A, B, C, are the angles of a triangle.

(1) $\cos A + \cos B + \cos C = 1 + 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$.

(2) $\cot \frac{A}{2} + \cot \frac{B}{2} + \cot \frac{C}{2} = \cot \frac{A}{2} \cot \frac{B}{2} \cot \frac{C}{2}$.

(3) $\cos A + \cos B - \cos C = 4 \cos \frac{A}{2} \cos \frac{B}{2} \sin \frac{C}{2} - 1$.

(4) $\tan \frac{B}{2} \{\sin A - \sin B + \sin C\} = 4 \sin \frac{B}{2} \sin \frac{C}{2} \sin \frac{A}{2}$.

(5) $\sin^2 A + \sin^2 B + \sin^2 C$
$= 2 \sin B \sin C \cos A + 2 \sin C \sin A \cos B + 2 \sin A \sin B \cos C$.

(6) $\cot A + \cot B + \cot C$
$= \cot A \cot B \cot C - \operatorname{cosec} A \operatorname{cosec} B \operatorname{cosec} C$.

(7) $\cos A + \cos B + \sin C = 4 \cos\left(45° - \frac{A}{2}\right) \cos\left(45° - \frac{B}{2}\right) \sin \frac{C}{2}$.

(8) $\cos \frac{A}{2} + \cos \frac{B}{2} + \cos \frac{C}{2}$
$= 4 \cos\left(\frac{180° - A}{4}\right) \cos\left(\frac{180° - B}{4}\right) \cos\left(\frac{180° - C}{4}\right)$.

(9) $\cos A + \sin B + \sin C$
$= 4 \cos \frac{A}{2} \cos\left(45° - \frac{B}{2}\right) \cos\left(45° - \frac{C}{2}\right) - 1$.

(10) $\sin (A + 2B) + \sin (B + 2C) + \sin (C + 2A)$
$= 4 \sin \frac{A - B}{2} \sin \frac{B - C}{2} \sin \frac{C - A}{2}$.

(11) $\sin^2 A + \sin^2 B + \sin^2 C = 2 + 2 \cos A \cos B \cos C$.

(12) $\cos A \operatorname{cosec} B . \operatorname{cosec} C + \cos B \operatorname{cosec} C \operatorname{cosec} A$
$+ \cos C \operatorname{cosec} A \operatorname{cosec} B = 2$.

(13) $\cot A + \frac{\sin A}{\sin B \sin C} = \cot B + \frac{\sin B}{\sin C \sin A}$.

(14) If in a triangle $\sin C = 2 \cos B \sin A$, prove that it is isosceles.

CHAPTER XVIII.

FORMULÆ FOR THE DIVISION OF ANGLES.

100. To prove that

$$2 \sin \frac{A}{2} = \pm \sqrt{1 + \sin A} \pm \sqrt{1 - \sin A}.$$

$$2 \cos \frac{A}{2} = \pm \sqrt{1 + \sin A} \mp \sqrt{1 - \sin A}.$$

We know that $\sin^2 \frac{A}{2} + \cos^2 \frac{A}{2} = 1,$

also that $2 \sin \frac{A}{2} \cos \frac{A}{2} = \sin A;$

adding, $\sin^2 \frac{A}{2} + 2 \sin \frac{A}{2} \cos \frac{A}{2} + \cos^2 \frac{A}{2} = 1 + \sin A,$

whence $\sin \frac{A}{2} + \cos \frac{A}{2} = \pm \sqrt{1 + \sin A}$ . . . (1)

subtracting, $\sin^2 \frac{A}{2} - 2 \sin \frac{A}{2} \cos \frac{A}{2} + \cos^2 \frac{A}{2} = 1 - \sin A,$

whence $\sin \frac{A}{2} - \cos \frac{A}{2} = \pm \sqrt{1 - \sin A}$ . . . (2)

adding (1) and (2),

$$2 \sin \frac{A}{2} = \pm \sqrt{1 + \sin A} \pm \sqrt{1 - \sin A},$$

subtracting (2) from (1),

$$2 \cos \frac{A}{2} = \pm \sqrt{1 + \sin A} \mp \sqrt{1 - \sin A}.$$

101. In order to find out which signs to give to the right-hand members of these expressions, we may proceed as follows :—

$$\sin\frac{A}{2}+\cos\frac{A}{2}=\sqrt{2}\left(\sin\frac{A}{2}\cdot\frac{1}{\sqrt{2}}+\cos\frac{A}{2}\cdot\frac{1}{\sqrt{2}}\right)=\sqrt{2}\sin\left(\frac{A}{2}+45^\circ\right).$$

$$\sin\frac{A}{2}-\cos\frac{A}{2}=\sqrt{2}\left(\sin\frac{A}{2}\cdot\frac{1}{\sqrt{2}}-\cos\frac{A}{2}\cdot\frac{1}{\sqrt{2}}\right)=\sqrt{2}\sin\left(\frac{A}{2}-45^\circ\right).$$

If the limits between which A must lie are given, we know the sign of $\sin\left(\frac{A}{2}+45^\circ\right)$; therefore we also know the sign which must be given to $\sqrt{1+\sin A}$. Similarly we can find the sign of $\sqrt{1-\sin A}$, which is the same as that of $\sin\left(\frac{A}{2}-45^\circ\right)$.

Example 1.—Show that $2\cos\frac{\theta}{2}=\sqrt{(1+\sin\theta)}-\sqrt{(1-\sin\theta)}$ when θ lies between $\frac{9\pi}{2}$ and $\frac{11\pi}{2}$.

Here $\frac{\theta}{2}+\frac{\pi}{4}$ lies between $\frac{5\pi}{2}$ and 3π,

$\frac{\theta}{2}-\frac{\pi}{4}$ " 2π " $\frac{5\pi}{2}$.

Between these limits $\sin\left(\frac{\theta}{2}+\frac{\pi}{4}\right)$ and $\sin\left(\frac{\theta}{2}-\frac{\pi}{4}\right)$ are both positive. Therefore $\sin\frac{\theta}{2}+\cos\frac{\theta}{2}$ and $\sin\frac{\theta}{2}-\cos\frac{\theta}{2}$ have positive values.

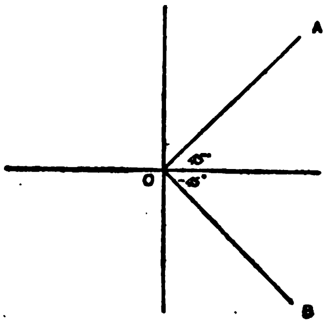
$$\therefore \sin\frac{\theta}{2}+\cos\frac{\theta}{2}=\sqrt{1+\sin\theta}.$$

$$\sin\frac{\theta}{2}-\cos\frac{\theta}{2}=\sqrt{1-\sin\theta}.$$

Subtracting, we obtain the required result.

Example 2.—Determine the limits between which $\frac{A}{2}$ must lie in order that

$$2\sin\frac{A}{2}=\sqrt{(1+\sin A)}-\sqrt{(1-\sin A)}.$$



Here we may argue simply as follows:—

$$\sin\frac{A}{2} + \cos\frac{A}{2} = \sqrt{1 + \sin A}.$$

$$\sin\frac{A}{2} - \cos\frac{A}{2} = -\sqrt{1 - \sin A}.$$

So the sign of the right-hand side depends on $\cos\frac{A}{2}$; it is positive when $\cos\frac{A}{2}$ is positive, negative when $\cos\frac{A}{2}$ is negative. Therefore $\cos\frac{A}{2}$ must be positive and greater than $\sin\frac{A}{2}$.

This is only the case when the bounding line of the angle lies between the bisectors of the first and fourth quadrant, viz. when $\frac{A}{2}$ lies between $n.360° + 45°$ and $n.360° - 45°$, where n is any integer.

EXAMPLES XVIII.

(1) Given $A = 330°$, find $\sin\frac{A}{2}$.

(2) Evaluate $\cos 202\frac{1}{2}°$.

(3) Find the sine and cosine of $101° 15'$.

Find $\sin A$ and $\cos A$ in terms of $\sin 2A$ when $2A$ lies between

(4) $90°$ and $270°$.

(5) $270°$ and $450°$.

(6) $990°$ and $1170°$.

(7) -630° and -720°.
(8) -450° and -540°.
(9) -720° and -810°.

Within what limits must $\frac{A}{2}$ lie in the following cases:—

(10) $2\cos\frac{A}{2} = \sqrt{1+\sin A} - \sqrt{1-\sin A}$.

(11) $2\cos\frac{A}{2} = -\sqrt{1+\sin A} + \sqrt{1-\sin A}$.

(12) $2\sin\frac{A}{2} = -\sqrt{1+\sin A} + \sqrt{1-\sin A}$.

MISCELLANEOUS EXAMPLES.

(1) Prove that $\cos 2(\beta-\gamma) + \cos 2(\gamma-\alpha) + \cos 2(\alpha-\beta) = 4\cos(\alpha-\beta)\cos(\beta-\gamma)\cos(\gamma-\alpha) - 1$.

(2) Show that $16\cos 6^\circ \cos 42^\circ \cos 66^\circ \cos 78^\circ = 1$.

(3) Find the value of $\sin\frac{A}{2}$ in terms of $\sin A$ when A lies between 900° and 990°.

(4) Solve the equation, $\cos x + \cos 3x = \frac{1}{2}$.

(5) If $a\tan\theta = b$, a and b being positive and θ between $\frac{5\pi}{4}$ and $\frac{3\pi}{2}$, find the values of $\sin 4\theta$ and $\cos\frac{\theta}{2}$.

(6) Show by a diagram that when $\cos\theta$ has a given value, $\cos\frac{\theta}{2}$ has two possible values.

(7) Prove that if $A + B + C = 0$,
$$\sin^2 A = \sin^2 B + \sin^2 C + 2\cos A\sin B\sin C.$$

(8) Prove that $\tan 15^\circ + \tan 75^\circ + \tan 135^\circ = 3$.

(9) What values of θ satisfy simultaneously the following equations:—

$$\sec\theta = -2, \quad \cot\theta = \frac{1}{\sqrt{3}}.$$

(10) Evaluate $\tan(45^\circ + A)\tan(135^\circ + A)$.

(11) Find $\cos(A + B)$ when $\cos A = \frac{7}{25}$, $\sin B = \frac{40}{41}$.

(12) Solve the equation, $\sin x + \cos x = \sqrt{2}\cos x$.

(13) Given $\sin A = \frac{2}{3}$ and A between 90° and 180°, find $\sin\frac{A}{2}$.

(14) The cosines of two angles of a triangle are $\frac{5}{13}$ and $\frac{7}{25}$ respectively; find the tangent of the third angle.

(15) Show that $\cos 12^\circ + \cos 108^\circ + \cos 132^\circ = 0$.

(16) If $\tan A$ be given, and a and a' are the corresponding values of $\tan\frac{A}{2}$, show that $a a' = -1$.

(17) Solve the equation,

$$\sin(x + a) + \cos(x + a) = \sin(x - a) + \cos(x - a).$$

(18) If $A + B + C = 0$, then

$$\cot A + \cot B + \cot C = \cot A \cot B \cot C - \operatorname{cosec} A \operatorname{cosec} B \operatorname{cosec} C.$$

(19) Prove that $\cos\left(\theta + \frac{\pi}{6}\right) - \cos\left(\theta + \frac{5\pi}{6}\right) = \sqrt{3}\cos\theta$.

(20) Trace the changes in sign and magnitude of $\sec 2A$ as A varies from 0 to 270°.

(21) When $A + B + C = 90^\circ$, show that

$$\tan A \tan B + \tan B \tan C + \tan C \tan A = 1.$$

(22) If $\sec\theta\tan\theta = 2\sqrt{3}$, find θ.

(23) Prove that $\sin 2(\beta - \gamma) + \sin 2(\gamma - a) + \sin 2(a - \beta) = -4\sin(\beta - \gamma)\sin(\gamma - a)\sin(a - \beta)$.

(24) Determine the correct signs in the formula for finding $\sin\frac{A}{2}$ when A lies between 810° and 990°.

(25) If $A + B + C = 0$, then

$$\sin (B + C) + \sin (C + A) - \sin (A + B) = 4 \sin \frac{C}{2} \cos \frac{A}{2} \cos \frac{B}{2}.$$

(26) Given, in a triangle, $\sin A = \frac{3}{5}$, $\cos B = \frac{24}{25}$, find $\cos C$.

(27) If $\cos a\theta + \cos b\theta = 0$, show that the various possible values of θ form two Arithmetical Progressions.

(28) If A and B are acute angles whose sum is greater than 90°, show that $\tan A \tan B > 1$.

(29) Show that

$$\sin 66^\circ \sin 6^\circ + \cos 48^\circ \cos 12^\circ + \cos 75^\circ \cos 15^\circ = 1.$$

(30) C and D are points on a circle on the same side of the diameter A B. Prove that $AC \cdot BD + AB \cdot CD = AD \cdot BC$.

(31) If $A + B + C = 180^\circ$, then

$$\cot \frac{180^\circ + A}{4} \cot \frac{180^\circ + B}{4} + \cot \frac{180^\circ + B}{4} \cot \frac{180^\circ + C}{4} + \cot \frac{180^\circ + C}{4} \cot \frac{180^\circ + A}{4} = 1.$$

(32) Solve the equation,

$$\cos 4\theta + 2 \cos \frac{3\theta}{2} \cos \frac{\theta}{2} = \cos \theta.$$

(33) Trace the changes of sign in $\sin x + \frac{\sqrt{3} - 1}{2} \cos x$, where x varies from 0° to 360°.

(34) If $\sin A = -\frac{24}{25}$, and A lies between 270° and 360°, find $\sin \frac{A}{2}$ and $\cos \frac{A}{2}$.

(35) Show that

$$\tan \left(m\pi + \frac{\pi}{12}\right) + \tan \left(n\pi + \frac{5\pi}{12}\right) + \tan \left(r\pi + \frac{3\pi}{4}\right) = 3,$$

m, n, r, being integers.

(36) If $A + B + C = 0$, prove that

$$\tan (A - C) + \tan (B - C) + \tan 3C = \tan (A - C) \tan (B - C) \tan 3C.$$

(37) If $A + B + C = 0$, prove that

$$\cos^2 A + \cos^2 B - \cos^2 C = 1 + 2 \sin A \sin B \cos C.$$

(38) Trace the changes in value of $\dfrac{\sin\theta}{1+\cos\theta}$ as θ varies from $(2n+1)\pi$ to $(2n+5)\pi$.

(39) Prove that if $A + B + C = 720°$,

$$\sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2} = 4\sin\frac{A}{4}\sin\frac{B}{4}\sin\frac{C}{4}.$$

(40) If $a\sin^2 x + b\sin x\cos x + c\cos^2 x = d$,

and $$\tan 2x = \frac{b}{c-a},$$

show that $$b^2 + (c-a)^2 = (2d - a - c)^2.$$

(41) If $\sin(\alpha - \beta)$, $\sin\alpha$, and $\sin(\alpha+\beta)$ are in Harmonical Progression, prove that $\sin^2\alpha = 2\cos^2\dfrac{\beta}{2}$.

(42) Find the least value of $\dfrac{\sqrt{1+\sin A}}{\sqrt{1-\sin A}} + \dfrac{\sqrt{1-\sin A}}{\sqrt{1+\sin A}}$.

PART IV.

CHAPTER XIX.

LOGARITHMS.

102. Given the following table:

$2^1 = 2$	$2^6 = 64$	$2^{11} = 2048$	$2^{16} = 65536$
$2^2 = 4$	$2^7 = 128$	$2^{12} = 4096$	$2^{17} = 131072$
$2^3 = 8$	$2^8 = 256$	$2^{13} = 8192$	$2^{18} = 262144$
$2^4 = 16$	$2^9 = 512$	$2^{14} = 16384$	$2^{19} = 524288$
$2^5 = 32$	$2^{10} = 1024$	$2^{15} = 32768$	$2^{20} = 1048576$

We can perform the *multiplication* of any powers of 2 within these limits by **addition** only, *division* by **subtraction** only, *raising to a given power* (or involution) by **multiplication**, *extracting a root* (or evolution) by **division.**

Example 1.—**Multiplication.**

$$8 \times 256 \times 512 = 2^3 \times 2^8 \times 2^9 = 2^{3+8+9} = 2^{20} = 1048576.$$

Example 2.—**Division.**

$$524288 \div 2048 = 2^{19} \div 2^{11} = 2^{19-11} = 2^8 = 256.$$

Example 3.—**Involution.**

$$8^6 = (2^3)^6 = 2^{3\times 6} = 2^{18} = 262144.$$

Example 4.—**Evolution.**

$$\sqrt[5]{32768} = \sqrt[5]{2^{15}} = 2^{\frac{15}{5}} = 2^3 = 8.$$

EXAMPLES XIX (a).

Using the table given above find the values of the following.

(1) $524288 \div 512$.

(2) $\sqrt[3]{262144}$.

(3) $512 \times 131072 \div 16384$.

(4) $(262144)^{\frac{5}{6}}$.

(5) $\sqrt[4]{16384 \times 262144}$.

(6) $\sqrt[7]{2^{13} \times 2^{14} \times 2^{15} \times 2^{17} \div 2^{10}}$.

(7) Assuming that money put out at compound interest doubles itself every 16 years, find the value 240 years hence of 5 shillings so invested.

(8) A blacksmith suggests charging one farthing for the first nail put in a horse's shoes, one halfpenny for the second, one penny for the third, twopence for the fourth, and so on, doubling the amount for each successive nail. What would be the charge for the nineteenth nail?

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103. The extension of the above processes to all numbers is the object of logarithms.

The calculated tables which enable us to apply them to practical purposes are called **Logarithmic tables.**

104. *Definition.*—If $a^x = m$, x is called the **logarithm of m to the base a**; in other words "the logarithm of a number to a given base is the index of the power to which that base must be raised to make it equal to the number."

This is written $x = \log_a m$.

Note 1.—The logarithm of unity to any base is 0.

$$\text{For} \quad a^0 = 1$$
$$\therefore \log_a 1 = 0.$$

whatever a may be.

Note 2.—The logarithm of the base is unity.

$$\text{For} \quad a^1 = a$$
$$\therefore \log_a a = 1.$$

Example 1.—Find the logarithm of 243 to base 3.

Here $3^x = 243 = 3^5$

$$\therefore x = 5.$$

Example 2.—Find $\log_4 2$.

That is find the logarithm of 2 to the base 4.

Here $4^x = 2$

or $\quad 2^{2x} = 2$

$$\therefore 2x = 1, \text{ or } x = \frac{1}{2}.$$

Example 3.—Find $\log_{2\sqrt{2}} 256$, that is, the logarithm of 256 to the base $2\sqrt{2}$.

Here $\quad (2\sqrt{2})^x = 256$

or $\quad \left(2^{\frac{3}{2}}\right)^x = 256$

that is $2^{\frac{3x}{2}} = 2^8$

$$\therefore \frac{3x}{2} = 8 \quad x = \frac{16}{3}.$$

Example 4.—Find the base when $\log 8 = \frac{3}{2}$;

that is when the logarithm of 8 is $\frac{3}{2}$

Here $x^{\frac{3}{2}} = 8.$

$$\therefore x = 8^{\frac{2}{3}} = \sqrt[3]{8^2} = 4.$$

Example 5.—Find the base when $\log 24\sqrt{3} = 3$.

Here $x^3 = 24\sqrt{3}$

$$x = (24\sqrt{3})^{\frac{1}{3}} = (8\sqrt{27})^{\frac{1}{3}} = 2\sqrt{3}.$$

EXAMPLES XIX (*b*).

Find the logarithms

(1) to base 2 of 8, 32, 256, 512.

(2) to base 3 of 9, 81, 729.

(3) to base 4 of 8, 16, 32, 128, 512.

(4) to base $2\sqrt{2}$ of 2, 16, 64, 128, 512.

(5) to base $3\sqrt{3}$ of 3, 27, 81, 243.

Find the base when—

(6) $\log 9 = 2$.

(7) $\log 27 = \frac{3}{2}$.

(8) $\log 8 = 2$

(9) $\log 125 = 6$.

(10) $\log 144 = 4$.

(11) $\log \sqrt{7} = \frac{1}{2}$.

(12) $\log 2 = \frac{2}{3}$.

(13) $\log 9 = 1 \cdot 5$.

---

105. Theorem 1.

The logarithm of the product of two or more numbers is equal to the sum of the logarithms of each of the numbers.

(1) Let there be two numbers m and n, and let a be the base.

We have to show that

$$\log_a m\,n = \log_a m + \log_a n.$$

Let $m = a^x,\ n = a^y \quad \therefore\ m\,n = a^{x+y}$,

then by the *definition of a logarithm*

$$\log_a m\,n = x + y = \log_a m + \log_a n.$$

(2) If there be more than two factors in the product, as m, n, p—

$$\log_a mnp = \log_a (m\,n) \times p = \log_a m\,n + \log_a p \text{ by (1)}$$
$$= \log_a m + \log_a n + \log_a p.$$

In the same way the theorem can be extended to any number of factors.

106. Theorem 2.

The logarithm of a quotient is equal to the logarithm of the numerator minus the logarithm of the denominator.

Let $\frac{m}{n}$ be the quotient, a the base; we have to show that

$$\log_a \frac{m}{n} = \log_a m - \log_a n.$$

Let $\qquad m = a^x, n = a^y \quad \therefore \frac{m}{n} = a^{x-y};$

then, by the *definition of a logarithm*,

$$\log_a \frac{m}{n} = x - y = \log_a m - \log_a n.$$

So we should have

$$\log_a \frac{m\,n}{p\,q} = \log_a m\,n - \log_a p\,q = \log_a m + \log_a n - \log_a p - \log_a q.$$

107. Theorem 3.

The logarithm of any power of a number is equal to the logarithm of the number multiplied by the index of the power.

Let m be any number, a the base; we have to show that

$$\log_a m^r = r \log_a m.$$

Let $\qquad m = a^x;$

raise each side to the power r, and where r is perfectly unrestricted in sign or magnitude;

$$\therefore m^r = a^{rx};$$

$\therefore$ by the *definition of a logarithm*

$$\log_a m^r = r\,x = r \log_a m.$$

Example 1.—Simplify

$$10 \log_a \frac{3}{2} + 7 \log_a \frac{5}{18} + 4 \log_a \frac{48}{25}$$

$$10 \log_a \frac{3}{2} = 10 \log_a 3 - 10 \log_a 2. \quad . \quad . \quad . \quad . \quad (1)$$

$$7 \log_a \frac{5}{18} = 7 \log_a 5 - 7 \log_a 18$$

$$= 7 \log_a 5 - 7 \log_a (2 \times 3^2)$$

$$= 7 \log_a 5 - 7 \log_a 2 - 14 \log_a 3. \quad . \quad (2)$$

$$4 \log_a \frac{48}{25} = 4 \log_a 16 \times 3 - 4 \log_a 5^2$$

$$= 4 \log_a 2^4 + 4 \log_a 3 - 8 \log_a 5$$

$$= 16 \log_a 2 + 4 \log_a 3 - \log_a 4 \quad . \quad . \quad (3)$$

adding (1), (2), and (3)

$$10 \log_a \frac{3}{2} + 7 \log_a \frac{5}{18} + 4 \log_a \frac{48}{25} = (10 - 14 + 4) \log_a 3$$

$$+ (-10 - 7 + 16) \log_a 2 + (2 + 7 - 8) \log_a 5$$

$$= - \log_a 5 - \log_a 2 = - \log_a 10.$$

Example 2.—Express in elementary form

$$\log \sqrt{\frac{a\,b^3}{(a-b)\,c^5}}$$

(the base, being the same throughout, is immaterial),

$$\log \sqrt{\frac{a\,b^3}{(a-b)\,c^5}} = \frac{1}{2} \log \frac{a\,b^3}{(a-b)\,c^5}$$

$$= \frac{1}{2} \{\log a + \log b^3 - \log (a-b) - \log c^5\}$$

$$= \frac{1}{2} \{\log a + 3 \log b - \log (a-b) - 5 \log c\}.$$

Example 3.—Given $\log_{10} 2 = \cdot 30103$ and $\log_{10} 3 = \cdot 4771213$, find the logarithm to base 10 of $\sqrt[3]{\frac{243}{128}}$.

Here

$$\log \sqrt[3]{\frac{243}{128}} = \frac{1}{3}\{\log 243 - \log 128\}$$
$$= \frac{1}{3}\{\log 3^5 - \log 2^7\}$$
$$= \frac{1}{3}\{5 \log 3 - 7 \log 2\}$$
$$= \frac{1}{3}\{2{\cdot}3856065 - 2{\cdot}10721\}$$
$$= \frac{1}{3} \text{ of } {\cdot}2783965$$
$$= {\cdot}0927988.$$

108. Theorem 4.

To find the relation between the *logarithms* of the *same number* to *different bases.*

Let m be the number, a and b two bases; it is required to find the relation between $\log_a m$ and $\log_b m$. Denote them by x and y respectively, then $a^x = b^y = m$,

$$\therefore \quad x = \log_a m,\ y = \log_b m$$

Now, since
$$a^x = b^y$$
$$a^{\frac{x}{y}} = b;$$
$$\therefore \quad \frac{x}{y} = \log_a b;$$
$$\therefore \quad x = y \log_a b,$$
that is
$$\mathbf{\log_a m = \log_b m \times \log_a b;}$$
similarly
$$\mathbf{\log_b m = \log_a m \times \log_b a.}$$

Example 1.—Find the relation between the logarithm of a number to a base, and the logarithm to the base raised to any power, *i.e.*, find the relation between the logarithm of m to the base a and the logarithm of m to the base a^r.

Here
$$\log_a m = \log_{a^r} m \times \log_a a^r$$
$$= \log_{a^r} m \times r \log_a a$$
$$= r \log_{a^r} m \text{ (since } \log_a a = 1 \text{ as in § 104).}$$

Thus $\quad \log_{10} m = 3 \log_{1000} m.$

Example 2.—To show that

$$\log_a b \times \log_b c \times \log_c a = 1.$$

Let $\log_a b = x, \log_b c = y, \log_c a = z.$

$\therefore \quad b = a^x, c = b^y, a = c^z.$

Now $c = b^y = (a^x)^y = a^{xy},$

so $a = c^z = (a^{xy})^z = a^{xyz},$

$$\therefore xyz = 1.$$

EXAMPLES XIX (*c*).

Express the following logarithms in their most elementary form.

(1) $\log a^{\frac{3}{2}} b^{\frac{2}{3}}.$

(2) $\log x^a y^b z^{-c}.$

(3) $\log \sqrt[3]{\dfrac{a x^2}{y^3}}.$

(4) $\log \sqrt[4]{\dfrac{a^2 x^3 y^{-2}}{b^4 z^{-1}}}.$

(5) $\log \dfrac{(a^2-x^2)^3}{(a+x)^{\frac{2}{3}}}.$

(6) $\log 225.$

(7) $\log \sqrt{72}.$

(8) $\log \dfrac{1}{4} \sqrt[3]{441}.$

(9) $\log \left(\dfrac{245 \times 128 \times 81}{121 \times 143}\right).$

(10) $\log \sqrt[3]{\dfrac{3\sqrt[3]{138}}{\sqrt[5]{\cdot 01}}}.$

Given $\log_{10}2 = \cdot 30103$, $\log_{10}3 = \cdot 4771213$, $\log_{10}7 = \cdot 8450980$, find the logarithms to the base 10 of the following.

(11) 128.

(12) 252.

(13) 294.

(14) $3 \cdot 6.$

(15) $124 \cdot 4.$

(16) $\sqrt{216}.$

(17) $\sqrt[3]{\dfrac{49}{16}}.$

(18) $\sqrt{\dfrac{224}{27}}.$

(19) $\sqrt[4]{\dfrac{7\sqrt{2}}{\sqrt{3}}}.$

(20) $\sqrt[3]{\dfrac{7 \cdot 2 \times 3\sqrt{7}}{\cdot 7 \times 2\sqrt{2}}}.$

(21) $3 \sin 60°.$

(22) $\tan 60°.$

(23) $\operatorname{cosec} 45°.$

(24) $\sec^2 60°.$

CHAPTER XX.

LOGARITHMS—*continued.*

109. Common Logarithms, or those in ordinary use, are **calculated to the base 10.**

Note.—When no special mention is made of the base it is assumed to be 10, thus log 7 means $\log_{10}7$.

110. From the following table it is clear that **all powers of 10 have integral logarithms,** and that successive powers of 10 which are in Geometrical Progression have logarithms which are in Arithmetical Progression.

$$
\begin{aligned}
\log 10000 &= \log 10^4 = 4,\\
\log 1000 &= \log 10^3 = 3,\\
\log 100 &= \log 10^2 = 2,\\
\log 10 &= \log 10^1 = 1,\\
\log 1 &= \log 10^0 = 0,\\
\log \frac{1}{10} &= \log 10^{-1} = -1,\\
\log \frac{1}{100} &= \log 10^{-2} = -2,\\
\log \frac{1}{1000} &= \log 10^{-3} = -3,\\
\log \frac{1}{10000} &= \log 10^{-4} = -4.
\end{aligned}
$$

111. Any number therefore which is **not an exact power of 10** must have a logarithm **intermediate in value** between the logarithms of the powers of 10 between which it lies. For example, 157 lies between 10^2 and 10^3, therefore its logarithm must be 2 + some fraction. This fraction is calculated in

Mathematical Tables to seven places of decimals and is obviously only an approximation, as no power of 10 is equal to 157 exactly.

112. *Definition.*—The *integral part* of any logarithm is called the **characteristic**, the *decimal part* the **mantissa.**

113. In our work we often come across a negative logarithm: for instance, $\log \frac{1}{2} = \log 1 - \log 2 = 0 - \cdot 3010300$. But it is found convenient **always to make the mantissa positive,** and to have a negative characteristic if necessary:

thus $-\cdot 3010300 = -1 + \cdot 6989700$, which is written $\bar{1}\cdot 6989700$,

so $-2\cdot 4982753 = -3 + \cdot 5017247$, which is written $\bar{3}\cdot 5017247$.

114. The characteristic can always be determined by inspection.

1. *When the number is greater than unity*, suppose the integral part of it has n figures, then it lies between 10^{n-1} and 10^n; so its logarithm lies between $n-1$ and n. That is, the characteristic is $n-1$, or **one less than the number of figures in the integral part of the given number.**
2. *When the number is less than unity*, suppose it a decimal with n cyphers immediately following the decimal point, then it must lie between $\frac{1}{10^n}$ and $\frac{1}{10^{n+1}}$, that is, between 10^{-n} and $10^{-(n+1)}$. Therefore the logarithm lies between $-n$ and $-(n+1)$, and is $-(n+1)+$ some fraction; so, remembering § 113, we see that the characteristic is negative and **greater by unity than the number of cyphers at the beginning of the decimal.**

Examples.—The characteristic of the logarithm of $2734\cdot 257$ is 3, and the characteristic of the logarithm of $\cdot 0000368$ is $\bar{5}$.

115. The mantissa of a logarithm depends only on the sequence of figures contained in the given number, and not at all on the value which those figures represent: thus the logarithms of 2583400, of $25\cdot 834$, of $\cdot 00025834$, have all the same mantissa, and differ only in their characteristic. This will be made clear by the following theorem.

116. Theorem.—When the logarithm of any number is given, we can determine the logarithm of the same number multiplied or divided by any power of 10.

Let N be the number, and let $\log N = a$.

Then
$$\log (N \times 10^m) = \log N + \log 10^m$$
$$= a + m.$$
$$\log (N \div 10^p) = \log N - \log 10^p$$
$$= a - p.$$

We see, therefore, that if we multiply or divide a number by an integral power of 10, the *mantissa* will remain unaltered, and there will only be a change in the characteristic.

Example 1.—Given $\log 2 = \cdot 30103$, find $\log 20000$ and $\log \cdot 00002$.

$$\log 20000 = \log (2 \times 10^4)$$
$$= \log 2 + 4 \log 10$$
$$= \log 2 + 4$$
$$= 4 \cdot 30103.$$
$$\log \cdot 00002 = \log (2 \div 10^5)$$
$$= \log 2 - \log 10^5$$
$$= \bar{5} \cdot 30103.$$

Example 2.—Given $\log 6506 \cdot 8 = 3 \cdot 8133675$, find $\log 65 \cdot 068$.

$$65 \cdot 068 = 6506 \cdot 8 \div 100.$$
$$\therefore \quad \log 65 \cdot 068 = \log 6506 \cdot 8 - \log 10^2$$
$$= 3 \cdot 8133675 - 2$$
$$= 1 \cdot 8133675.$$

117. In Mathematical Tables the digits of the *mantissæ* alone are given, and we must supply the characteristic. For instance, in the tables we find opposite the number 59391 the figures 7737206, which means that the mantissa is $\cdot 7737206$; but the number lies between 10000 (10^4) and 100000 (10^5). $\therefore$ the characteristic is 4. $\therefore$ $\log 59391 = 4 \cdot 7737206$.

118. When a logarithm with negative characteristic has to be divided by a number which is not an exact divisor of the characteristic, we have to proceed as follows in order to keep the characteristic integral:—

Suppose the logarithm $\bar{2}\cdot30103$ has to be divided by 7. Write it $-7+5\cdot30103$, and then perform the division. This gives

$$-1+\cdot75729,$$

or,

$$\bar{1}\cdot75729.$$

Examples.—Find $\sqrt[5]{\cdot6}$.

We find

$$\log 6 = \cdot7781513.$$

$$\therefore\ \log\cdot6 = \bar{1}\cdot7781513.$$

$$\log\sqrt[5]{\cdot6} = \frac{1}{5}\log\cdot6 = \frac{\bar{1}\cdot7781513}{5}$$

$$= \frac{-5+4\cdot7781513}{5}$$

$$= \bar{1}\cdot9556302.$$

From the tables we find that $\cdot9556302$ is the logarithm of $9\cdot028804$.

$$\therefore\ \bar{1}\cdot9556302 = \log\cdot9028804.$$

$$\therefore\ \sqrt[5]{6} = \cdot9028804.$$

EXAMPLES XX.

(1) What are the characteristics of the logarithms of $287\cdot5$, $\cdot259$, $\cdot0037$, $2\cdot63$, $\cdot00045$, 38, 10^{4}, $\sqrt[3]{3895}$, $\sqrt[2]{(687)^3}$.

(2) Subtract $\bar{2}\cdot83472$ from $\bar{4}\cdot10427$, and divide each of them by 5.

(3) Multiply $\bar{3}\cdot2598105$ by 4, and divide the result by 9.

(4) Given $\log 15\cdot652 = 1\cdot1945698$, write down the logarithms of 156250, $1565\cdot2$, $\cdot0015652$, $\frac{1}{\cdot15652}$, and $\frac{1}{156\cdot52}$.

(5) Given $\log\cdot040573 = \bar{2}\cdot6082371$, write down the logarithms of 40573, $\cdot40573$, $\sqrt{405\cdot73}$, $\frac{1}{40\cdot573}$, and $\frac{1}{\sqrt[3]{4\cdot0573}}$.

Given log 3 = ·4771213, log 4 = ·6020600, log 7 = ·8450980, log 11 = 1·0413927, find the logarithms of

(6) ·21.

(7) ·077.

(8) ·22.

(9) ·015.

(10) ·00231.

(11) $\dfrac{1}{\sqrt{1\cdot 1}}$.

(12) $\dfrac{3}{\sqrt[3]{84}}$.

(13) $\sqrt[3]{\dfrac{4}{\sqrt{7}}}$.

(14) $4\sqrt{\cdot 14}$.

(15) $\sqrt[5]{\dfrac{11}{\sqrt{147}}}$.

(16) sin 45°.

(17) tan 30°.

(18) cos 30°.

(19) $\sqrt{\sin 30^\circ \sin 60^\circ}$.

(20) $\sqrt[3]{\dfrac{\cot 60^\circ \tan 30^\circ}{\sec 45^\circ}}$.

Given the same four logarithms, find x in the following cases to two decimal places:—

(21) $3^x = 7$.

(22) $11^{2x-1} = 12^4$.

(23) $4^{1-x} = 21^{1+x}$.

(24) $3^{2x} \times 5^{3x-4} = 7^{x-1} \times 11^{2-x}$.

CHAPTER XXI.

EXPLANATION OF MATHEMATICAL TABLES.

119. It will be assumed throughout the present chapter that when a number is increased by a small quantity the **increase of the logarithm is proportional to the increase of the number.** For instance, that if the logarithm of 22613 is $4\cdot3543582$ and that of 22614 is $4\cdot3543774$, the logarithm of $22613\cdot5$ will be midway between these two values, viz. $4\cdot3543678$. This is called the **Principle of Proportional Parts.**

The proof requires the application of methods beyond the scope of this book.

120. Trigonometrical Ratios are numbers, therefore their logarithms follow the ordinary rules of other numbers; but as all sines and cosines and many tangents and cotangents are less than unity, and therefore have their logarithms negative, it is usual to add 10 to them and they are printed in the Tables in that form.

The symbol L is used for any logarithm with 10 added, thus $L \sin 20°$ means $10 + \log \sin 20°$. In the Tables we find $L \sin 20° = 9\cdot5340517$;

$$\therefore \log \sin 20° = 9\cdot5340517 - 10 = -\cdot4659483.$$

121. In Books of Tables the logarithms of all the natural numbers are given from 1 to 99999 and logarithms of trigonometrical ratios of the angles between 0° and 90° increasing by a difference of 1′; thus log 87563 and $L \sin 35° 41'$ can be found directly. These logarithms are calculated to 7 places of decimals. Such books also contain the actual values (called the "natural values") of the trigonometrical ratios of all angles from 0° to 90°: we can look out in them, for instance, the value of the cosine of 13° 27′, and find it to be $\cdot9725733$.

Explanations of the Tables are always given in the books themselves, so we give here examples only of the four principal purposes for which they are used.

122. (1) **To find the logarithm of any seven figured number.**

Find log 4436728.

From the tables

$$\begin{aligned} \log 4436700 &= 6 \cdot 6460601 \\ \log 4436800 &= 6 \cdot 6460699 \\ \hline \text{difference for } 100 &= \quad \cdot 0000098 \\ \text{Let difference for } 28 &= \qquad x \end{aligned}$$

$$\text{then } \frac{x}{\cdot 0000098} = \frac{28}{100}$$

$$\begin{aligned} \text{whence } x &= \cdot 0000027, \\ \therefore \log 4436728 &= 6 \cdot 6460601 \\ &+ \quad \cdot 0000027 \\ &= 6 \cdot 6460628 \end{aligned}$$

Note.—The work is entirely independent of the position of the decimal point in the number. Thus, had we been asked to find the logarithm of 44·36728 we should have proceeded as above quite regardless of the decimal point, and made the characteristic in the result 1 instead of 6. If we had been asked for the logarithm of ·004436728 we should similarly have found it to be $\bar{3} \cdot 6460628$ (see § 114).

123. (2) **Find the number corresponding to the logarithm 3·7678642** . . . . (α)

From the tables

$$\begin{aligned} \log 5859 \cdot 5 &= 3 \cdot 7678606 \quad . \quad . \quad . \quad (\beta) \\ \log 5859 \cdot 6 &= 3 \cdot 7678680 \\ \hline \text{difference for } \cdot 1 &= \quad \cdot 0000074 \\ \text{,,} \qquad x &= \quad \cdot 0000036 \end{aligned}$$

x being the difference between the number required and 5859·5 and ·0000036 being found by subtracting (β) from (α).

$$\therefore x = \frac{\cdot 1 \times \cdot 0000036}{\cdot 0000074} = \cdot 048;$$

$$\begin{aligned} \therefore \text{the number required} &= 5859 \cdot 5 \\ &+ \quad \cdot 048 \\ &= 5859 \cdot 548 \end{aligned}$$

124. Applications of these two methods.

(α) Find approximately the fifth root of 2639147.

Using tables, we find $\log 2639147 = 6 \cdot 4214635$.

$$\therefore \log \sqrt[5]{2639147} = \tfrac{1}{5}(6 \cdot 4214635) = 1 \cdot 2842927.$$

Then, as in § 123, we find that the number whose logarithm is $1 \cdot 2842927$ is $19 \cdot 24388$;

$$\therefore \sqrt[5]{2639147} \text{ is } 19 \cdot 24388.$$

(β) Find $\sqrt[7]{\cdot 07}$.

We find $\log \cdot 07 = \bar{2} \cdot 8450980$;

$$\begin{aligned} \therefore \log \sqrt[7]{\cdot 07} &= \tfrac{1}{7}(\bar{2} \cdot 8450980) \\ &= \tfrac{1}{7}(\bar{7} + 5 \cdot 8450980) \\ &= \bar{1} \cdot 8350140. \end{aligned}$$

Then from the tables we find that $\bar{1} \cdot 8350140$ is the logarithm of $\cdot 6839337$;

$$\therefore \sqrt[7]{\cdot 07} = \cdot 6839337.$$

EXAMPLES XXI (a).

(1) Given $\log 96513 = 4 \cdot 9845858$,
$\log 96514 = 4 \cdot 9845903$,
find $\log 96513 \cdot 35$.

(2) Given $\log 40621 = 4 \cdot 6087506$,
$\log 40622 = 4 \cdot 6087613$,
find $\log 40621 \cdot 27$.

(3) Given $\log 2 \cdot 3778 = \cdot 3761753$,
$\log 2 \cdot 3779 = \cdot 3761936$,
find $\log 237 \cdot 7854$.

(4) Given $\log \cdot 31710 = \bar{1} \cdot 5011962$,
$\log \cdot 31711 = \bar{1} \cdot 5012099$,
find $\log \cdot 03171058$.

(5) Given $\log \cdot 063516 = \bar{2}\cdot 8028831$,
$\log \cdot 063517 = \bar{2}\cdot 8028900$,
find $\log 6\cdot 351672$.

(6) Given $\log 79\cdot 564 = 1\cdot 9007166$,
*diff. = 55,
find $\log \cdot 007956464$.

(7) Given $\log 8\cdot 5782 = \cdot 9333962$,
diff. = 50,
find $\log 8578\cdot 236$.

(8) Given $\log 1564\cdot 5 = 3\cdot 1943756$.
diff. = 277,
find $\log \cdot 1564586$.

Find x in the following six cases (to 7 significant figures):—

(9) $\log x = 2\cdot 5650732$, given $\log 36734 = 4\cdot 5650682$,
$\log 36735 = 4\cdot 5650800$.

(10) $\log x = \bar{1}\cdot 9385782$, given $\log 868\cdot 11 = 2\cdot 9385748$,
$\log 868\cdot 12 = 2\cdot 9385798$.

(11) $\log x = 1\cdot 6694125$, given $\log \cdot 46710 = \bar{1}\cdot 6694099$,
$\log \cdot 46711 = \bar{1}\cdot 6694192$.

(12) $\log x = \bar{3}\cdot 2598372$, given $\log 18190 = 4\cdot 2598327$,
$\log 18191 = 4\cdot 2598566$.

(13) $\log x = \cdot 1437829$, given $\log \cdot 013924 = \bar{2}\cdot 1437640$,
diff. = 312.

(14) $\log x = 3\cdot 7243875$, given $\log \cdot 53014 = \bar{1}\cdot 7243906$,
diff. = 82.

(15) Given $\log 8\cdot 5762 = \cdot 9332949$,
$\log 5\cdot 5519 = \cdot 7444416$,
$\log 5\cdot 5518 = \cdot 7444338$,
find the number of figures in the value of $(85762)^{12}$, and calculate $(\cdot 00085762)^{\frac{1}{12}}$ to 7 places of decimals.

(16) Given $\log 299 = 2\cdot 4756712$,
$\log 300 = 2\cdot 4771213$,
$\log 2 = \cdot 30103$,
find the nearest integers to the value of $2^{\frac{281}{20}}$.

* Means that the difference between log 79564 and log 79565 = ·0000055.

(17) What will £1000 amount to if put out at compound interest for 10 years at 4 per cent.?

Given $\log 1 \cdot 04 = \cdot 0170333$, $\log \cdot 14802 = \bar{1} \cdot 1703204$,
$\log \cdot 14803 = \bar{1} \cdot 1703497$.

(18) What is the compound interest on £10,000 for 12 years at 5 per cent.?

Given $\log 105 = 2 \cdot 0211893$, $\log 1 \cdot 7958 = \cdot 2542580$,
$\log 1 \cdot 7959 = \cdot 2542822$.

(19) Given $\log 2 = \cdot 30103$, $\log 103 = 2 \cdot 0128372$, find how many years will elapse before a sum of money doubles itself at 3 per cent. per annum compound interest.

(20) Given $\log 3 = \cdot 4771213$, $\log 62403 = 4 \cdot 7952055$,
$\log 1 \cdot 38 = \cdot 1398791$, $\log 62404 = 4 \cdot 7952124$,

find the value of $\sqrt{\dfrac{3\sqrt[3]{138}}{\sqrt[5]{\cdot 01}}}$.

---

125. (3) **To find the logarithms of the Trigonometrical ratios of any angle.***

(α) To find L sin 42° 31′ 47″.

From the tables—

$$\text{L} \sin 42° 31' = 9 \cdot 8298212$$
$$\text{L} \sin 42° 32' = 9 \cdot 8299589.$$

$$\therefore \text{ difference for } 1' \text{ or } 60'' = \cdot 0001377,$$
$$\therefore \text{ difference for } 47'' = \tfrac{47}{60} \text{ of } \cdot 0001377$$
$$= \cdot 0001079.$$
$$\mathbf{L \sin 42° 31' 47'' = 9 \cdot 8299291.}$$

(β) To find L cot 37° 28′ 33″.

As before, from tables we find—

$$\text{difference for } 60'' = \cdot 0002616.$$
$$\therefore \text{ difference for } 33'' = \tfrac{33}{60} \text{ of } \cdot 0002616$$
$$= \cdot 0001439.$$

* Note that sines, tangents, and secants, and therefore their logarithms, increase as the angle increases; cosines, cotangents, cosecants, and their logarithms, decrease as the angle increases.

So	L cot 37° 28′	being	10·1155428
subtract for	33″		·0001439
	L cot 37° 28′ 33″	**=**	**10·1153989**

126. (4) **Given the logarithm of a Trigonometrical ratio, find the corresponding angle.**

(α) Find the angle which corresponds to the logarithmic tangent 9·8941987.

From the tables we find—

$$\text{L}\tan 38^\circ\, 5' = 9{\cdot}8941114;\ \text{diff. for } 1' = {\cdot}0002601.$$

but $$\text{L}\tan 38^\circ\, 5'\, x'' = 9{\cdot}8941987$$

$$\therefore\ \text{diff. for } x'' = {\cdot}0000873$$

$$\therefore\ x = \frac{873}{2601} \times 60'' = 20'';$$

$\therefore$ the required angle is **38° 5′ 20″.**

(β) Find the angle which corresponds to the logarithmic cosine 9·9428976.

From the Tables—

$$\text{L}\cos 28^\circ\, 44' = 9{\cdot}9429335;\ \text{diff. for } 1' = -\ {\cdot}0000692$$

but $$\text{L}\cos 28^\circ\, 44'\, x'' = 9{\cdot}9428976$$

$$\therefore\ \text{diff. for } x'' = -\ {\cdot}0000359$$

$$\therefore\ x = \frac{359}{692} \times 60'' = 31'';$$

$\therefore$ the required angle is **28° 44′ 31″.**

127. Natural Sines, Cosines, &c., are the values furnished in Mathematical Tables of all Trigonometrical Ratios of angles between 0° and 90°, ascending by a difference of 1′.

These values are approximations given correct to 7 places of decimals, except in the isolated cases which can be expressed as terminating decimals.

Example.— $\sin 30^\circ = \dfrac{1}{2} = {\cdot}5,$

$$\cos 30^\circ = \frac{\sqrt{3}}{2} = {\cdot}8660254.$$

The principle of Proportional Parts holds for the natural Trigonometrical Ratios, and enables us to find the angles correct to tenths of seconds, when the values of the Ratios are given or *vice versâ*.

Example 1.—Find sin 35° 14′ 43·5″.

From the tables—

$$\sin 35^\circ 14' = .5769076$$
$$\sin 35^\circ 15' = \cdot 5771452$$
$$\therefore \text{ difference for } 60'' = \cdot 0002376$$
$$\therefore \text{ difference for } 43\cdot 5'' = \tfrac{43\cdot 5}{60} \text{ of } \cdot 0002376$$
$$= \cdot 0001722.$$
$$\therefore \sin 35^\circ 14' 43\cdot 5'' = \cdot 5770798.$$

Example 2.—Given the natural cosine of an angle to be ·3245671, to find the angle.

From the tables—

$$\cos 71^\circ 3' = \cdot 3247429, \left\{\begin{array}{l}\text{diff. for } 1' \text{ or } 60'' \\ = -\cdot 0002751.\end{array}\right.$$
$$\cos 71^\circ 3' x'' = \cdot 3245671$$
$$\text{diff. for } x'' = -\cdot 0001758$$
$$\therefore \quad x = \frac{1758}{2751} \times 60'' = 38\cdot 3''.$$

$\therefore$ the required angle is **71° 3′ 38·3″**.

EXAMPLES XXI (*b*).

(1) Given L cos 53° 48′ = 9·7712976,
L cos 53° 49′ = 9·7711249,
find L cos 53° 48′ 46″.

(2) Given L sin 22° 29′ = 9·5825345,
diff. for 1′ = 3052,
find L sin 22° 29′ 36″.

(3) Given L tan 38° 11′ = 9·8956719,
diff. for 1′ = 2600,
find L tan 38° 11′ 27″.

(4) Given $$L \operatorname{cosec} 42^\circ 32' = 10{\cdot}1700411,$$ $$L \operatorname{cosec} 42^\circ 33' = 10{\cdot}1699034,$$

find $$L \operatorname{cosec} 42^\circ 32' 15''.$$

(5) Given $$L \cos 81^\circ 20' = 9{\cdot}1780721,$$ $$\text{diff. for } 1' = 8296,$$

find $$L \cos 81^\circ 21' 50''.$$

(6) Given $$L \tan 61^\circ 24' = 10{\cdot}2634301,$$ $$\text{diff. for } 1' = 3006,$$

find the angle whose logarithmic tangent is $10{\cdot}2636209$.

(7) Given $$L \cot 30^\circ 16' = 10{\cdot}2339051,$$ $$L \cot 30^\circ 17' = 10{\cdot}2336149,$$

find the angle whose L cot is $10{\cdot}2337594$.

(8) Given $$L \sin 72^\circ 40' = 9{\cdot}9798158,$$ $$\text{diff.} = 394,$$

find the angle whose L sin is $9{\cdot}9798348$.

(9) Given $$\cos 55^\circ 20' = {\cdot}5688011,$$ $$\cos 55^\circ 21' = {\cdot}5685619,$$

find $$\cos 55^\circ 20' 25''.$$

(10) Given $$\operatorname{cosec} 18^\circ 4' = 3{\cdot}2245230,$$ $$\text{diff. for } 1' = 28727,$$

find $$\operatorname{cosec} 18^\circ 4' 10''.$$

(11) Find the value of $\sqrt{\dfrac{\sin A}{\sin B}}$, where

$$A = 42^\circ 14' 32'' \quad . \quad L \sin 42^\circ 14' = 9{\cdot}8274671$$
$$L \sin B = 9{\cdot}3985762 \quad . \quad L \sin 42^\circ 15' = 9{\cdot}8276063$$
$$\log 16386 = 4{\cdot}2144730,$$
$$\log 16387 = 4{\cdot}2144995.$$

(12) Find the value of $\sqrt[3]{16 \cos^2 A}$, given

$$A = 11^\circ 37' 18'' \qquad L \cos 11^\circ 37' = 9{\cdot}9910119$$
$$\log 2 = {\cdot}30103 \qquad \text{diff. for } 1' = 260$$
$$\log 24852 = 4{\cdot}3953613$$
$$\text{diff. for } 1 = 175$$

CHAPTER XXII.

PROPERTIES CONNECTING THE TRIGONOMETRICAL RATIOS OF THE ANGLES OF A TRIANGLE WITH THE SIDES.

N.B.—In all cases the lengths of the sides opposite the angles A, B, C, are denoted by a, b, c.

128. **I. In any triangle the sides are proportional to the sines of the opposite angles**; *i.e.* $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$.

Fig. 1.

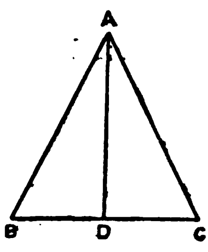
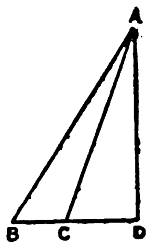


Fig. 2.



From one of the angular points, A, of a triangle A B C, draw A D perpendicular to B C, produced if necessary.

Then $$AD = AC \sin ACD,$$

also $$AD = AB \sin ABD.$$

$$\therefore \quad AC \sin ACD = AB \sin ABD,$$

that is $$b \sin C = c \sin B$$

(since A C D is C in Fig. 1, and the supplement of C in Fig. 2).

$$\therefore \quad \frac{b}{\sin B} = \frac{c}{\sin C}.$$

In a similar way, by drawing a perpendicular from C on the opposite side, we can show that $\frac{a}{\sin A} = \frac{b}{\sin B}$.

$$\therefore \quad \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \quad . \quad . \quad . \quad (\alpha)$$

If the triangle has a right angle at C, as in Fig. 3, it is obvious that

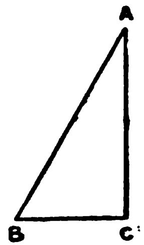
$$\sin B = \frac{b}{c}, \text{ and } \sin A = \frac{a}{c}.$$

$$\therefore \quad c = \frac{b}{\sin B} = \frac{a}{\sin A},$$

which is the same result as (α), since in this case $\sin C = 1$.

The theorem is therefore universally proved.

Fig. 3.



129. II. In any triangle A B C, show that

$$\mathbf{a = b \cos C + c \cos B} \quad . \quad . \quad . \quad . \quad (\beta)$$
$$\mathbf{b = c \cos A + a \cos C} \quad . \quad . \quad . \quad . \quad (\gamma)$$
$$\mathbf{c = a \cos B + b \cos A} \quad . \quad . \quad . \quad . \quad (\delta)$$

Using the figures of the preceding section, we have—

(1) When C is an acute angle

$$BC = BD + DC,$$

but $\frac{BD}{b} = \cos C \qquad \frac{DC}{c} = \cos B;$

$$\therefore \quad a = b \cos C + c \cos B.$$

(2) When C is an obtuse angle

$$BC = BD - CD,$$

but

$$\frac{BD}{c} = \cos B, \; \frac{CD}{b} = \cos ACD = \cos(180° - C) = -\cos C;$$

$$\therefore \quad CD = c \cos B, \text{ and } \therefore \quad CD = -b \cos C.$$

$$\therefore \quad a = c \cos B + b \cos C.$$

(3) When C is a right angle,

$$BC = AB \cos B,$$

or, $$a = c \cos B,$$
$$= c \cos B + b \cos C;$$

since $$\cos C = \cos 90° = 0.$$

The theorem therefore holds universally.

In a similar way, by drawing perpendiculars from B and C on the opposite sides, formulæ (γ) and (δ) can be proved.

130. These last results, and many others, can be deduced at once from the relations

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \quad . \quad . \quad . \quad . \quad (\alpha)$$

(1) To deduce

$$a = b \cos C + c \cos B,$$

we know that

$$\sin A = \sin (B + C),$$

viz., that

$$\sin A = \sin B \cos C + \cos B \sin C. \quad . \quad . \quad . \quad (\beta')$$

From (α) we may suppose

$$\left.\begin{array}{l} \sin A = a k \\ \sin B = b k \\ \sin C = c k \end{array}\right\} \text{ where } k \text{ is a constant.}$$

Substituting these values in (β'), we get

$$k a = k b \cos C + k c \cos B,$$

or, $$a = b \cos C + c \cos B.$$

(2) Prove that

$$\frac{a^2 + b^2}{a^2 + c^2} = \frac{1 + \cos (A - B) \cos C}{1 + \cos (A - C) \cos B}.$$

From (α) we obtain by ordinary algebraical methods

$$\frac{a^2 + b^2}{a^2 + c^2} = \frac{\sin^2 A + \sin^2 B}{\sin^2 A + \sin^2 C}.$$

Then

$$\frac{2\sin^2 A + 2\sin^2 B}{2\sin^2 A + 2\sin^2 C} = \frac{1 - \cos 2A + 1 - \cos 2B}{1 - \cos 2A + 1 - \cos 2C}$$

$$= \frac{2 - (\cos 2A + \cos 2B)}{2 - (\cos 2A + \cos 2C)} = \frac{2 - 2\cos(A+B)\cos(A-B)}{2 - 2\cos(A+C)\cos(A-C)}$$

$$= \frac{1 + \cos(A-B)\cos C}{1 + \cos(A-C)\cos B} \quad \begin{pmatrix}\text{since} & \cos(A+B) = -\cos C \\ \text{and} & \cos(A+C) = -\cos B\end{pmatrix},$$

which establishes the required relation.

EXAMPLES XXII (*a*).

Prove the following relations in a triangle :—

(1) $\dfrac{\sin^2 A + \sin B \sin C}{\sin^2 A - \sin B \sin C} = \dfrac{a^2 + bc}{a^2 - bc}$.

(2) $a \cos \dfrac{B - C}{2} = (b + c) \sin \dfrac{A}{2}$.

(3) $\dfrac{a}{b - c} = \dfrac{\sin \dfrac{B + C}{2}}{\sin \dfrac{B - C}{2}}$.

(4) $(a + b + c) \sin \dfrac{A}{2} = 2a \cos \dfrac{B}{2} \cos \dfrac{C}{2}$.

(5) $(a - b)(1 + \cos C) = c(\cos B - \cos A)$.

(6) $(b^2 - c^2) \sin A = a^2 \sin(B - C)$.

(7) $ab \sin C(\cot A - \cot B) = b^2 - a^2$.

(8) $a \cos B - b \cos A = c \operatorname{cosec}^2 C(\sin^2 A - \sin^2 B)$.

(9) $a \cos A + b \cos B + c \cos C =$
$2a \cos B \cos C + 2b \cos C \cos A + 2c \cos A \cos B$.

(10) $a \sin A + b \sin B + c \sin C =$
$\sqrt{a^2 + b^2 + c^2} \cdot \sqrt{\sin^2 A + \sin^2 B + \sin^2 C}$.

(11) If a, b, c be in Arithmetical Progression, prove that cosec $(B + C)$, cosec $(C + A)$, cosec $(A + B)$ are in Harmonical Progression.

(12) If $\frac{\cos A}{\cos B} = \frac{p}{q}$, show that $\cos C = \frac{a\,p - b\,q}{b\,p - a\,q}$.

(13) If D be any angle, prove that

$$a \sin (D - B) + b \sin (D + A) = c \sin D.$$

(14) Also that $b \cos (D - C) + c \cos (D + B) = a \cos D$.

(15) If $A : B : C :: 2 : 3 : 4$, then $\frac{a + c}{2\,b} = \cos \frac{A}{2}$.

(16) If a, b, c be in Arithmetical Progression show that

$$\cot \frac{C}{2} = 3 \tan \frac{A}{2}.$$

(17) Prove that $\cos A + \cos B + \cos C$

$$= \sin \frac{A}{2} \cos \frac{B - C}{2} + \sin \frac{B}{2} \cos \frac{C - A}{2} + \sin \frac{C}{2} \cos \frac{A - B}{2}.$$

---

131. III. To show that in any triangle A B C

$$\mathbf{c^2 = a^2 + b^2 - 2\,a\,b \cos C} \quad . \quad . \quad . \quad (\lambda)$$
$$\mathbf{a^2 = b^2 + c^2 - 2\,b\,c \cos A} \quad . \quad . \quad . \quad (\mu)$$
$$\mathbf{b^2 = c^2 + a^2 - 2\,c\,a \cos B} \quad . \quad . \quad . \quad (\nu)$$

(1) When C is an acute angle,

$$AB^2 = BC^2 + AC^2 - 2\,BC \cdot DC; \text{ (Euc. ii. 13)}$$

but $$\frac{DC}{AC} = \cos C; \quad \therefore DC = b \cos C,$$

or $$c^2 = a^2 + b^2 - 2\,a\,b \cos C.$$

Fig. 4.

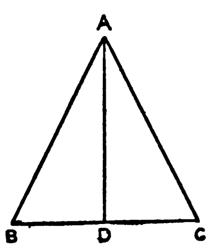
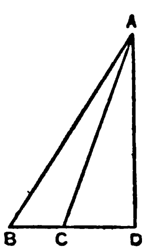


Fig. 5.



(2) When C is an obtuse angle,

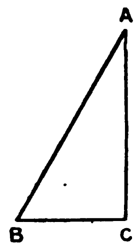
$$AB^2 = BC^2 + AC^2 + 2\,BC \cdot CD; \text{ (Euc. ii. 12)}$$

but $$\frac{CD}{AC} = \cos ACD = \cos(180^\circ - C) = -\cos C;$$

$$\therefore CD = -b\cos C,$$

or $$c^2 = a^2 + b^2 - 2ab\cos C.$$

Fig. 6.



(3) When C is a right angle,

$$AB^2 = BC^2 + AC^2; \text{ (Euc. i. 47)}$$

or $$c^2 = a^2 + b^2 = a^2 + b^2 - 2ab\cos C,$$

since $$\cos C = \cos 90^\circ = 0;$$

$\therefore$ the theorem holds universally.

(μ) and (ν) can be proved similarly.

Note.—(β), (γ), (δ), and (λ), (μ), (ν), are not independent sets of formulæ, but either series may be deduced from the other as follows:—

(1) Adding (μ) and (ν) we obtain

$$a^2 + b^2 = a^2 + b^2 + 2c^2 - 2bc\cos A - 2ca\cos B;$$

or $$2c^2 = 2bc\cos A + 2ca\cos B;$$

or $$c = b\cos A + a\cos B.$$

(2) Multiplying (δ) by c, (γ) by b, (β) by a, we obtain]

$$c^2 = bc\cos A + ac\cos B,$$
$$b^2 = ab\cos C + bc\cos A,$$
$$a^2 = ac\cos B + ab\cos C,$$

subtracting the last two results from the first

$$c^2 - b^2 - a^2 = -2ab\cos C;$$

or $$c^2 = b^2 + a^2 - 2ab\cos C.$$

132. From the formulæ (μ), (ν), and (λ) we obtain

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}, \cos B = \frac{c^2 + a^2 - b^2}{2ca}, \cos C = \frac{a^2 + b^2 - c^2}{2ab},$$

which enable us to find the angles of any triangle when the sides are given.

Example.—Find A, B, and C when

$$a = 4, b = 2\sqrt{2}, c = 2 + 2\sqrt{3}.$$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{8 + 4 + 8\sqrt{3} + 12 - 16}{8\sqrt{2}(\sqrt{3}+1)}$$

$$= \frac{8 + 8\sqrt{3}}{8\sqrt{2}(\sqrt{3}+1)} = \frac{1}{\sqrt{2}}; \quad \therefore A = 45°.$$

$$\cos B = \frac{c^2 + a^2 - b^2}{2ca} = \frac{4 + 8\sqrt{3} + 12 + 16 - 8}{16(\sqrt{3}+1)}$$

$$= \frac{24 + 8\sqrt{3}}{16(\sqrt{3}+1)} = \frac{\sqrt{3}}{2}; \quad \therefore B = 30°.$$

whence $\qquad C = 180° - 30° - 45° = 105°.$

133. **IV. To express the cosine, sine, and tangent of the half-angles of a triangle in terms of the sides.**

$$(1) \cos A = \frac{b^2 + c^2 - a^2}{2bc};$$

$$\therefore 1 + \cos A = \frac{2bc + b^2 + c^2 - a^2}{2bc}$$

$$= \frac{(b+c)^2 - a^2}{2bc} = \frac{(b+c+a)(b+c-a)}{2bc}.$$

Let $a + b + c$, the perimeter of the triangle, be denoted by $2s$, then $b + c - a = (2s - 2a)$;

$$\therefore 1 + \cos A = 2\cos^2\frac{A}{2} = \frac{2s(2s - 2a)}{2bc};$$

$$\therefore \cos^2\frac{A}{2} = \frac{s(s-a)}{bc}; \quad \text{or, } \cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}.$$

(2) Again—

$$1 - \cos A = 1 - \frac{b^2 + c^2 - a^2}{2bc} = \frac{2bc - b^2 - c^2 + a^2}{2bc}$$

$$= \frac{a^2 - (b-c)^2}{2bc} = \frac{(a+c-b)(a+b-c)}{2bc};$$

$$\therefore 1 - \cos A = 2\sin^2\frac{A}{2} = \frac{(2s-2b)(2s-2c)}{2bc};$$

$$\therefore \sin^2\frac{A}{2} = \frac{(s-b)(s-c)}{bc}; \text{ or, } \sin\frac{A}{2} = \frac{\sqrt{(s-b)(s-c)}}{bc}.$$

$$(3)\ \frac{\sin\frac{A}{2}}{\cos\frac{A}{2}} = \frac{\sqrt{(s-b)(s-c)}}{\sqrt{bc}} \div \frac{\sqrt{s(s-a)}}{\sqrt{bc}};$$

$$\therefore \tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}.$$

Note.—The values of $\cos\frac{B}{2}$, $\sin\frac{B}{2}$, &c., can be deduced from those of $\cos\frac{A}{2}$, $\cos\frac{A}{2}$, &c., by writing B for A, b for a, c for b, a for c.

$$s = \frac{a+b+c}{2}, \text{ which will remain unaltered.}$$

The changes indicated are called **cyclic changes**, a quantity like s which remains unchanged is called **symmetrical.**

134. V. **To express the sine of an angle of a triangle in terms of the sides.**

$$\sin A = 2\sin\frac{A}{2}\cdot\cos\frac{A}{2}$$

$$= 2\cdot\frac{\sqrt{(s-b)(s-c)}}{\sqrt{bc}} \times \frac{\sqrt{s(s-a)}}{\sqrt{bc}} \text{ from (2) and (1)}$$

$$= \frac{2\sqrt{s(s-a)(s-b)(s-c)}}{bc}.$$

EXAMPLES XXII (*b*).

Prove that in a triangle

(1) $b\cos^2\frac{C}{2}+c\cos^2\frac{B}{2}=\frac{a+b+c}{2}$.

(2) $\frac{b^2-c^2}{a}\cos A+\frac{c^2-a^2}{b}\cos B+\frac{a^2-b^2}{c}\cos C=0$.

(3) $(b+c-a)\tan\frac{A}{2}=(c+a-b)\tan\frac{B}{2}=(a+b-c)\tan\frac{C}{2}$.

(4) $\tan^2\frac{A}{2}\tan^2\frac{B}{2}\tan^2\frac{C}{2}=\left(1-\frac{a}{s}\right)\left(1-\frac{b}{s}\right)\left(1-\frac{c}{s}\right)$.

(5) $(a+b+c)(\cos A+\cos B+\cos C)=$
$$2a\cos^2\frac{A}{2}+2b\cos^2\frac{B}{2}+2c\cos^2\frac{C}{2}.$$

(6) $c^2\cos^2 B+b^2\cos^2 C+bc\cos(B-C)=$
$$bc\cos A+ca\cos B+ab\cos C.$$

(7) Find A when $b=2$, $c=3$, $a=\sqrt{7}$.

(8) Find C when $a=2\sqrt{2}$, $b=3$, $c=\sqrt{5}$.

(9) Find $\tan\frac{B}{2}$ when $a=4$, $b=8$, $c=6$.

(10) Find $\sin\frac{C}{2}$ when $a=3$, $b=5$, $c=6$.

(11) Find sin A when $a=4$, $b=2\cdot4$, $c=3\cdot2$.

(12) Find sin A when $a=24$, $b=70$, $c=74$.

(13) Find all the angles when $a=\sqrt{6}$, $b=\sqrt{3}+1$, $c=2$.

(14) Find the greatest angle of a triangle when the sides are $\sqrt{2}$, b, and $\sqrt{4+2b+b^2}$.

(15) If $(a^2+b^2)\cos 2A=b^2-a^2$, prove that the triangle is right angled.

(16) If D be the middle point of BC and the angle $B>C$, prove that
$$\cot ADB=\frac{b^2-c^2}{2ac\sin B}.$$

(17) $\tan\dfrac{A}{2} = \dfrac{5}{6}$, $\tan\dfrac{B}{2} = \dfrac{20}{37}$; find $\tan C$ and show that $a + c = 2b$.

(18) If $\cos A + \cos B + \cos C = 4$, then $a + b = 4c$.

(19) If the bisector of the angle B A C meet B C in D, then $AD = \dfrac{2bc}{b+c}\cos\dfrac{A}{2}$.

(20) If $A - B = 90°$, show that $2c^{-2} = (a+b)^{-2} + (a-b)^{-2}$.

(21) If $a : b : c :: 2m : m^2 - 1 : m^2 + 1$, show that $C = 90°$.

(22) If

$$(2+\sqrt{3})(a+c-b)(a+b-c) = (2-\sqrt{3})(a+b+c)(b+c-a),$$

show that $A = 30°$.

(23) From the relation $a^2 = b^2 + c^2 - 2bc\cos A$, prove that a is less than $b + c$.

(24) From the vertices of a triangle A B C perpendiculars A D, B E, C F are drawn on the opposite sides. Show that D E, E F, F D are $c\cos C$, $a\cos A$, $b\cos B$ respectively.

CHAPTER XXIII.

SOLUTION OF TRIANGLES.

135. We have shown in Chapter VI. how to solve **right-angled triangles** with certain data; the examples there given required only very simple calculations.

For more complicated cases it is best to use logarithms in our work, thus substituting addition for multiplication, and subtraction for division. For instance, if we had to find c from the formula $c = \dfrac{a}{\sin A}$, where $a = 237 \cdot 42$ and $A = 32° \; 17'$, the division of $237 \cdot 42$ by $\cdot 5341065$ would be laborious: with the use of logarithmic tables our work could be simplified as follows:—

$$\begin{aligned} \log c = \log a - \log \sin A &= \log a - L \sin A + 10 \\ &= 2 \cdot 3755173 - 9 \cdot 7276278 + 10 \\ &= 2 \cdot 6478895, \end{aligned}$$

and we find that the number whose log is $2 \cdot 6478895$ is $444 \cdot 518$; that is $\mathbf{c = 444 \cdot 518}$.

Example 1.—Given $c = 312 \cdot 4768$, $A = 37° \; 41' \; 36''$, solve the triangle (right-angled at C).

With the help of tables, we find

$$\begin{aligned} \log 312 \cdot 4768 &= 2 \cdot 4948177, \\ L \sin 37° \; 41' \; 36'' &= 9 \cdot 7863502, \\ L \cos 37° \; 41' \; 36'' &= 9 \cdot 8983383. \end{aligned}$$

Then (1) $a = c \sin A$.

$$\begin{aligned} \therefore \quad \log a = \log c + \log \sin A &= \log c + L \sin A - 10 \\ &= 2 \cdot 4948177 + 9 \cdot 7863502 - 10 \\ &= 2 \cdot 2811679, \end{aligned}$$

and so from the tables $\quad \mathbf{a = 191 \cdot 0592.}$

(2) $b = c \cos A$.

$$\therefore \quad \log b = \log c + \mathrm{L} \cos A - 10$$
$$= 2 \cdot 4948177 + 9 \cdot 8983383 - 10$$
$$= 2 \cdot 3931560,$$

and so from the tables $\quad$ **b = 247·2612.**

(3) B is of course the complement of A, and so B = **52° 18′ 24″**.

Example 2.—Solve a right-angled triangle in which $a = 71 \cdot 829$, $b = 80 \cdot 918$ (C is the right angle).

(1) With the help of tables, we find

$$\log 71 \cdot 829 = 1 \cdot 8562998,$$
$$\log 80 \cdot 918 = 1 \cdot 9080451.$$

Then $\quad \tan A = \dfrac{a}{b},$

so, $\quad \mathrm{L} \tan A = \log a - \log b + 10$
$$= \cdot 8562998 - \cdot 9080451 + 10$$
$$= 9 \cdot 9482547.$$

Whence from tables, which give L tan 41° 35′ and L tan 41° 36′, we calculate **A** to be **41° 35′ 40·9″**.

(2) **B**, being the complement of A, is **48° 24′ 19·1″**.

(3) From tables, we calculate

$$\mathrm{L} \sin 41^\circ\, 35'\, 40 \cdot 9'' = 9 \cdot 8220745.$$

Then $\quad c = \dfrac{a}{\sin A},$

so, $\quad \log c = \log a - \mathrm{L} \sin A + 10$
$$= 1 \cdot 8562998 - 9 \cdot 8220745 + 10$$
$$= 2 \cdot 0342253$$

Whence we find $\quad$ **c = 108·1995.**

136. When two sides are given, whose ratio can be easily calculated, we can solve the triangle, without logarithms, by the use of tables of the natural values of the trigonometrical ratios (see § 127).

Example.—Solve a triangle, right-angled at C, in which $c = 4$, $a = 3 \cdot 4682$.

(1) Here

$$\sin A = \frac{a}{c} = \frac{3\cdot4682}{4} = \cdot86705,$$

and from the tables of natural sines which give $\sin 60° 7' = \cdot8670417$ and $\sin 60° 8' = \cdot8671866$, we calculate $\cdot86705$ to be the sine of $60° 7' 3\cdot4''$.

$$\therefore \quad \mathbf{A = 60° 7' 3\cdot4''}.$$

(2) We find $\cos 60° 7' 3\cdot4'' = \cdot49822$.

$$\therefore \quad \mathbf{b} = c \cos A = 4 \times \cdot49822 = \mathbf{1\cdot99288}.$$

(3) **B** (the complement of A) $= \mathbf{29° 52' 56\cdot6''}$.

EXAMPLES XXIII (*a*).

Find in a triangle, right-angled at C,

(1) the other parts when $c = 16\sqrt{3}$, $b = 24$.

(2) the other sides when $c = 18$, $A = 15°$.

(3) the angles when $a = 4\cdot72518$, $c = 6$;

given
$$\sin 51° 58' = \cdot7876524,$$
$$\sin 51° 57' = \cdot7874732.$$

(4) a and c when $b = 12\cdot73$, $A = 42°$;

given

$L \tan 42° = 9\cdot9544374$	$\log 11\cdot4621 = 1\cdot0592658$
$L \sin 42° = 9\cdot8255109$	$\log 1\cdot7129 = \cdot2337320$
$\log 12\cdot73 = 1\cdot1048284$	$\log 1\cdot7130 = \cdot2337574$

(5) The angles when $a = 12\cdot487$, $b = 7\cdot9824$;

given

$\log 12\cdot487 = 1\cdot0964581$	$L \tan 57° 25' = 10\cdot1944197$
$\log 7\cdot982 = \cdot9021117$	$L \tan 57° 24' = 10\cdot1941413$

(6) b when $a = 83\cdot45$, $c = 217\cdot54$;

given

$\log 300\cdot99 = 2\cdot4785521$	$\log 200\cdot9 = 2\cdot3029799$
$\log 134\cdot09 = 2\cdot1273964$	$\log 200\cdot89 = 2\cdot3029583$

What additional logarithms must be given in the following cases :—

(7) $c = 247 \cdot 85$, $B = 71° 24'$,

$\log 247 \cdot 85 = 2 \cdot 3941889$ $\quad$ $L \sin 71° 24' = 9 \cdot 9767022$

$L \cos 71° 24' = 9 \cdot 5037353$,

find a and b.

(8) $\log a = 3 \cdot 5965971$ $\quad$ $L \sin A = 9 \cdot 7692187$

$L \sin B = 9 \cdot 9079576$,

find b and c.

---

137. In the general solution of triangles, not right-angled, four cases will present themselves. We may be given—

(1) **one side and two angles ;**

(2) **two sides and the angle included by them ;**

(3) **three sides ;**

(4) **two sides and an angle opposite to one of them.**

138. **I. Given the side c, and two angles A and B, find C, a, and b.**

$C = 180° - (A + B)$, and is therefore also known.

Now $$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}. \quad (\S\ 128.)$$

$\therefore$ (1) $a = \dfrac{c \sin A}{\sin C}$, whence, if the calculations are simple, we at once find a; or taking logarithms

$$\log a = \log c + L \sin A - L \sin C,$$

and (2) $$b = \frac{c \sin B}{\sin C}.$$

or, $$\log b = \log c + L \sin B - L \sin C.$$

$\therefore$ C, a, and b are expressed in terms of the known quantities.

Example.—Given $c = 4315 \cdot 74$,

$B = 63° 21' 58''$,

$C = 51° 37' 24''$,

find A, a, and b.

Since $A = 180° - B - C$, we find at once that **A = 65° 0′ 38″**, and by calculation from tables, we find

$$\log 4315 \cdot 74 = 3 \cdot 6350552,$$
$$L \sin B = L \sin 63° \, 21' \, 58'' = 9 \cdot 9512837,$$
$$L \sin C = L \sin 51° \, 37' \, 24'' = 9 \cdot 8942863,$$
$$L \sin A = L \sin 65° \; 0' \, 38'' = 9 \cdot 9573130.$$

Then (1) $$\frac{a}{\sin A} = \frac{c}{\sin C}, \text{ or } a = \frac{c \sin A}{\sin C}.$$

$$\therefore \quad \log a = \log c + L \sin A - L \sin C.$$

$$\begin{array}{rr} & 3 \cdot 6350552 \\ + & 9 \cdot 9573130 \\ \hline & 13 \cdot 5923682 \\ - & 9 \cdot 8942863 \\ \hline \log a = & 3 \cdot 6980819 \end{array}$$

and so from tables **a = 4989·78.**

(2) Similarly $\log b = \log c + L \sin B - L \sin C$.

$$\begin{array}{rr} & 3 \cdot 6350552 \\ + & 9 \cdot 9512837 \\ \hline & 13 \cdot 5863389 \\ - & 9 \cdot 8942863 \\ \hline \log b = & 3 \cdot 6920526 \end{array}$$

and so from tables **b = 4921·01.**

EXAMPLES XXIII (*b*).

(1) $A = 30°$, $B = 45°$, $a = 6\sqrt{2}$, find b and C.

(2) $B = 60°$, $C = 75°$, $b = 2\sqrt{3}$, find a, c, and A.

(3) $A = 23° \, 35'$, $B = 120°$, $a = 8$, find b and C.
Given $\sin 23° \, 35' = \cdot 4$.

(4) $C = 64° \, 10'$, $A = 30°$, $c = 18$, find a and B.
Given $\sin 64° \, 10' = \cdot 9$.

(5) $B = 45°$, $C = 10°$, $a = 200$, find b.

Given $L \sin 55° = 9{\cdot}9133645$, $\log 1{\cdot}7264 = {\cdot}2371414$,

$\log 2 = {\cdot}3010300$, $\log 1{\cdot}7265 = {\cdot}2371666$,

(6) $B = 29°\ 17'$, $C = 135°$, $a = 123$, find c.

ven

$\log 123 = 2{\cdot}0899051$, $\log 3211 = 3{\cdot}5066403$,

$\log 2 = {\cdot}3010300$, diff. for 1 = 1352,

$L \sin 15°\ 43' = 9{\cdot}4327777$.

(7) $B = 26°\ 30'$, $C = 47°\ 15'$ $a = 1652$, solve the triangle.

Given

$L \sin 73°\ 45' = 9{\cdot}9822938$, $\log 7678 = 3{\cdot}8852481$,

$L \sin 47°\ 15' = 9{\cdot}8658868$, diff. = 57,

$L \sin 26°\ 30' = 9{\cdot}6495274$, $\log 12636 = 4{\cdot}1016096$,

$\log 1652 = 3{\cdot}2180100$, diff. = 344.

(8) $A = 89°\ 9'\ 24''$, $B = 54°\ 33'\ 25''$, $a = 15236$, solve the triangle.

Given

$L \sin 89°\ 9' = 9{\cdot}9999522$, $\log 12414 = 4{\cdot}0939116$,

$L \sin 89°\ 10' = 9{\cdot}9999541$, $\log 90179 = 4{\cdot}9551054$,

$L \sin B = 9{\cdot}9109936$, $\log 9018 = 3{\cdot}9551102$.

$L \sin (A + B) = 9{\cdot}7721909$,

$\log 15236 = 4{\cdot}1828710$,

(9) Two angles and the adjacent side of a triangle are

$\gamma - \alpha$, $\alpha - \beta$, and $a\,\{\cos(\alpha + \beta - \gamma) - \cos(\alpha - \beta + \gamma)\}$;

find the other sides and angle.

---

139. **II. Given the sides b, c, and the angle A, find the angles B, C, and the side a.**

Two methods of solution are applicable—

a. When the calculations are simple ones we can find a from the formula,

$$a^2 = b^2 + c^2 - 2\,b\,c \cos A,$$

and then find B or C from the relations $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$.

But the cosine formula is not suitable for logarithmic calculation, and so,

β. Generally, we proceed as follows:—

Since
$$\frac{\sin B}{\sin C} = \frac{b}{c} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

we have, subtracting unity from each side of (1),
$$\frac{\sin B - \sin C}{\sin C} = \frac{b-c}{c};$$

similarly adding unity to each side of (1),
$$\frac{\sin B + \sin C}{\sin C} = \frac{b+c}{c}.$$

By division,
$$\frac{\sin B - \sin C}{\sin B + \sin C} = \frac{b-c}{b+c},$$

or,
$$\frac{2\sin\frac{B-C}{2} \cdot \cos\frac{B+C}{2}}{2\cos\frac{B-C}{2} \cdot \sin\frac{B+C}{2}} = \frac{b-c}{b+c},$$

or,
$$\frac{\tan\frac{B-C}{2}}{\tan\frac{B+C}{2}} = \frac{b-c}{b+c}.$$

Hence
$$\tan\frac{B-C}{2} = \frac{b-c}{b+c} \cdot \tan\frac{B+C}{2},$$

or,
$$\tan\frac{\mathbf{B-C}}{\mathbf{2}} = \frac{\mathbf{b-c}}{\mathbf{b+c}}\cot\frac{\mathbf{A}}{\mathbf{2}} \quad . \quad . \quad . \qquad (2)$$

From (2) $\tan\frac{B-C}{2}$ can be found in terms of the known quantities, b, c, and A; therefore $\frac{B-C}{2}$ is known. And $\frac{B+C}{2}$ is known since $B + C = 180° - A$; therefore B and C are known.

To find a we can use the formula,
$$\frac{a}{\sin A} = \frac{b}{\sin B}. \quad \therefore a = b\,\frac{\sin A}{\sin B} \quad . \quad . \qquad (3)$$

Taking logarithms of (2) and (3) we obtain

$$\text{L} \tan \frac{B-C}{2} = \log (b-c) - \log (b+c) + \text{L} \cot \frac{A}{2} \quad . \quad (2)'$$

$$\log a = \log b + \text{L} \sin A - \text{L} \sin B \; . \quad . \quad . \quad . \quad (3)'$$

Note.—The following method is sometimes adopted to find a:—

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = \frac{b+c}{\sin B + \sin C};$$

$$\therefore \; a=(b+c)\frac{\sin A}{\sin B + \sin C} = \frac{(b+c) \cdot 2 \sin \frac{A}{2} \cos \frac{A}{2}}{2 \sin \frac{B+C}{2} \cdot \cos \frac{B-C}{2}}.$$

But $$\sin \frac{B+C}{2} = \cos \frac{A}{2};$$

$$\therefore \; a = \frac{(b+c) \sin \frac{A}{2}}{\cos \frac{B-C}{2}},$$

or $$\log a = \log (b+c) + \text{L} \sin \frac{A}{2} - \text{L} \cos \frac{B-C}{2}.$$

Example a.—Given $a = 2\sqrt{3}$, $b = 3 + \sqrt{3}$, $C = 60°$, solve the triangle.

Here

$$\begin{aligned} c^2 &= a^2 + b^2 - 2\,a\,b \cos C \\ &= 12 + 12 + 6\sqrt{3} - 2\sqrt{3}(3+\sqrt{3}) \quad [\text{since } 2 \cos 60° = 1] \\ &= 12 + 12 + 6\sqrt{3} - 6\sqrt{3} - 6 \\ &= 18. \end{aligned}$$

$\therefore$ **c = 3 √2.**

And $$\sin A = \frac{a \sin C}{c} = \frac{2\sqrt{3}}{3\sqrt{2}} \times \frac{\sqrt{3}}{2} = \frac{1}{\sqrt{2}}.$$

$$\therefore \; \mathbf{A = 45°.}$$

And $$B = (180° - A - C).$$

$$\therefore \; \mathbf{B = 75°.}$$

Example β.—Given $A = 56° \ 15' \ 47\cdot5''$, $b = 55634\cdot02$ ft., $c = 35388\cdot92$ ft., solve the triangle.

(1) With the help of tables we calculate—

$$\log(b - c) = \log 20245\cdot1 \quad = 4\cdot3063200,$$
$$\log(b + c) = \log 91022\cdot94 = 4\cdot9591509,$$

$$L \cot\frac{A}{2} = L \cot 28° \ 7' \ 53\cdot75'' = 10\cdot2719229.$$

$$\therefore L \tan\frac{B - C}{2} = L \cot\frac{A}{2} + \log(b - c) - \log(b + c) \text{ becomes}$$

$$= 10\cdot2719229 + 4\cdot30632 - 4\cdot9591509$$

$$= 9\cdot619092.$$

From tables we find this to be the value of

$$L \tan 22° \ 35' \ 14\cdot1''.$$

$$\therefore \frac{B - C}{2} = 22° \quad 35' \quad 14\cdot1''$$

also, $\dfrac{B + C}{2} = 90° - \dfrac{A}{2}$; $\quad \therefore \dfrac{B + C}{2} = 61° \quad 52' \quad 6\cdot25''$

$$B = 84° \ 27' \ 20\cdot35''$$
$$C = 39° \ 16' \ 52\cdot15''$$

(2) To find a we use the formula,

$$a = \frac{b \sin A}{\sin B},$$

$$\text{or else, } a = \frac{(b + c)\sin\dfrac{A}{2}}{\cos\dfrac{B - C}{2}}.$$

Taking the latter, we calculate from tables—

$$L \sin\frac{A}{2} = 9\cdot6734800,$$

$$L \cos\frac{B - C}{2} = 9\cdot9653408,$$

whence $\log a = \log(b+c) + \text{L}\sin\frac{\text{A}}{2} - \text{L}\cos\frac{\text{B}-\text{C}}{2}$

$= 4 \cdot 9591509 + 9 \cdot 6734800 - 9 \cdot 9653408$

$= 4 \cdot 6672901.$

So we find **a** = **46482·57.**

EXAMPLES XXIII (c).

(1) If $a = 3$, $b = 8$, $\text{C} = 60°$, find c.

(2) $a = \sqrt{3} + 1$, $b = \sqrt{3} - 1$, $\text{C} = 60°$, solve the triangle.

(3) Two sides of a triangle are 5 and 6 feet, and the included angle is 53° 8′, solve the triangle.
Given $\tan 26° 34' = \frac{1}{2}$; $\tan 10° 18' = \cdot 182$.

(4) $b = 12$, $c = 6$, $\text{A} = 70°$, find B and C.
Given
$\cot 35° = 1 \cdot 428148$, $\tan 25° 27' = \cdot 4759048$,
$\tan 25° 28'' = \cdot 4762616$.

(5) $c = 60$, $a = 48$, $\text{B} = 54° 36' 24''$, find the other angles.
Given
$\cot 27° 18' = 1 \cdot 9374645$, $\tan 12° 8' = \cdot 2149900$,
$\cot 27° 19' = 1 \cdot 9360825$, $\tan 12° 9' = \cdot 2152944$.

(6) $b = 128 \cdot 5$, $c = 27$, $\text{A} = 25° 30'$, find the other angles.
Given
$\log 2 \cdot 03 = \cdot 3074960$, $\text{L}\tan 70° 52' = 10 \cdot 4597547$,
$\log 3 \cdot 11 = \cdot 4927604$, diff. for $1' = 4082$,
$\text{L}\cot 12° 45' = 10 \cdot 6453598$.

(7) $a = 43$ ft., $b = 11$ ft., $\text{C} = 44°$, find A and B.
Given $\log 2 = \cdot 3010300$, $\text{L}\tan 55° 42' = 10 \cdot 1661177$,
$\log 3 = \cdot 4771213$, $\text{L}\tan 55° 43' = 10 \cdot 1663891$,
$\text{L}\cot 22° = 10 \cdot 3935904$.

(8) $b = 14$, $c = 11$, $A = 60°$, solve the triangle.

Given $\left.\begin{matrix}\log 2\\ \log 3\end{matrix}\right\}$ as above, $\log 1{\cdot}2767 = {\cdot}10609$

$L \tan 11° 44' = 9{\cdot}3174299$, $L \cos 11° 44' = 9{\cdot}9908291$,

$L \tan 11° 45' = 9{\cdot}3180640$, $L \cos 11° 45' = 9{\cdot}9908029$.

(9) $b = 131$, $c = 72$, $A = 40°$, find B and C.

Given

$\log 5{\cdot}9 = {\cdot}7708520$, $L \tan 38° 36' = 9{\cdot}9021604$,

$\log 2{\cdot}03 = {\cdot}3074960$, diff. for $1' = 2591$,

$L \cot 20° = 10{\cdot}4389341$.

(10) $a = 284$, $b = 482$, $C = 37°$, solve the triangle.

Given $\log 198 = 2{\cdot}2966652$, $L \tan 37° 41' = 9{\cdot}8878554$,

$\log 766 = 2{\cdot}8842288$, $L \tan 37° 42' = 9{\cdot}8881165$,

$L \cot 18° 30' = 10{\cdot}4754801$, $\log 284 = 2{\cdot}4533183$,

$L \sin 33° 48' 46'' = 9{\cdot}7455503$, $\left\{\begin{matrix}\log 307{\cdot}06 = 2{\cdot}4872232,\\ \text{diff.} = 141.\end{matrix}\right.$

$L \sin 37° = 9{\cdot}7794630$,

---

140. III. Given the sides, a, b, c, find the angles.

(α) When the calculations are simple we can use the formula $\cos A = \dfrac{b^2 + c^2 - a^2}{2\,b\,c}$, and either of the two similar ones as in § 132.

(β) When the triangle has to be worked by logarithms, we can use any of the formulæ for finding the half-angles, which are all well adapted for logarithmic calculations.

For example, from

$$\tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}},$$

we deduce at once

$$\log \tan \frac{A}{2} = \frac{1}{2}\{\log (s-b) + \log (s-c) - \log s - \log (s-a)\};$$

or,

$$L \tan \frac{A}{2} = 10 + \frac{1}{2}\{\log (s-b) + \log (s-c) - \log s - \log (s-a)\}.$$

Examples.—Given $a = 229{\cdot}464,$
$b = 152{\cdot}976,$
$c = 127{\cdot}48,$
solve the triangle.

Here we find $s = \dfrac{a+b+c}{2} = 254{\cdot}96.$

From tables we get—

$$\begin{aligned} \log s &= \log 254{\cdot}96 = 2{\cdot}4064721, \\ \log(s-a) &= \log 25{\cdot}496 = 1{\cdot}4064721, \\ \log(s-b) &= \log 101{\cdot}984 = 2{\cdot}0085321, \\ \log(s-c) &= \log 127{\cdot}48 = 2{\cdot}1054421. \end{aligned}$$

Then

$$\begin{aligned} \text{L}\tan\frac{A}{2} &= 10 + \frac{1}{2}\{\log(s-b) + \log(s-c) - \log s - \log(s-a)\}, \\ &= 10 + \frac{1}{2}\{4{\cdot}1139742 - 3{\cdot}8129442\}, \\ &= 10{\cdot}1505150. \end{aligned}$$

And so from tables, we calculate

$$\frac{A}{2} = 54^\circ\ 44'\ 8{\cdot}2'',$$

$$\mathbf{A = 109^\circ\ 28'\ 16{\cdot}4''}.$$

Similarly,

$$\begin{aligned} \text{L}\tan\frac{B}{2} &= 10 + \frac{1}{2}\{\log(s-c) + \log(s-a) - \log s - \log(s-b)\}, \\ &= 10 + \frac{1}{2}\{3{\cdot}5119142 - 4{\cdot}4150042\}, \\ &= 10 - {\cdot}4515450 = 9{\cdot}5484550, \end{aligned}$$

and so from tables

$$\frac{B}{2} = 19^\circ\ 28'\ 16{\cdot}4'',$$

$$\mathbf{B = 38^\circ\ 56'\ 32{\cdot}8''}.$$

$$\therefore\quad \mathbf{C} = (180^\circ - A - B) = \mathbf{31^\circ\ 35'\ 10{\cdot}8''}.$$

EXAMPLES XXIII (*d*).

(1) $a = 5, b = 7, c = 8$, find B.

(2) $a = 13, b = 5\sqrt{2}, c = 17$, find A.

(3) The sides of a triangle are a feet, b feet, and $\sqrt{a^2 + ab + b^2}$ feet in length, find the greatest angle.

(4) $a = 3, b = 6, c = 7$, find the smallest angle.
Given
$$\cos 25° 12' = \cdot 9048271,$$
$$\cos 25° 13' = \cdot 9047032.$$

(5) $a = 13, b = 40, c = 45$, find the angles.
Given
$$\tan 29° 46' = \frac{4}{7},\ \tan 52° 6' = \frac{9}{7},$$

(6) $a = 32, b = 40, c = 66$, find the greatest angle.
Given
$$\log 207 = 2\cdot 3159703, \qquad L \tan 23° 42' = 9\cdot 6424342,$$
$$\log 1073 = 3\cdot 0305997, \qquad L \tan 23° 43' = 9\cdot 6427773.$$

(7) $a = 131, b = 106, c = 75$, find A.
Given
$$\log 2 = \cdot 3010300, \qquad L \tan 45° 32' = 10\cdot 0080857,$$
$$\log 3 = \cdot 4771213, \qquad L \tan 45° 33' = 10\cdot 0083384,$$
$$\log 13 = 1\cdot 1139434.$$

(8) $a = 4, b = 5, c = 6$, find B.
Given
$$\log 2 = \cdot 30103, \qquad L \cos 27° 53' = 9\cdot 9464040,$$
$$L \cos 27° 54' = 9\cdot 9463371.$$

(9) $a = 60, b = 160, c = 180$, find all the angles.
Given
$$\log 2 = \cdot 3010300, \qquad L \tan 9° 35' = 9\cdot 2274706,$$
$$\log 7 = \cdot 8450980, \qquad \text{diff. for } 1' = 7689,$$
$$L \tan 30° 36' = 9\cdot 7718801, \text{ diff. for } 1' = 2883.$$

(10) $a = 275\cdot 35, b = 189\cdot 28, c = 301\cdot 47$, find A.
Given
$$\log 38305 = 4\cdot 5832555, \qquad \log 19377 = 4\cdot 2872865,$$
$$\log 10770 = 4\cdot 0322157, \qquad \log 8158 = 3\cdot 9115837,$$
$$L \tan 31° 45' = 9\cdot 7915635, \text{ diff. for } 1' = 2823.$$

141. IV. **Given two sides a and b, and the angle A opposite to one of them; find B, C, and c.**

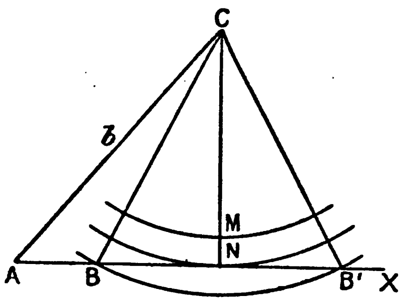
From $\frac{a}{\sin A} = \frac{b}{\sin B}$ we have $\sin B = \frac{b \sin A}{a}$; but if $b \sin A$ is greater than a, $\sin B$ will be greater than unity, which is impossible; therefore, to make the solution possible, $b \sin A$ must either be equal to a or less than a.

(1) When $b \sin A = a$, $\sin B = 1 \quad \therefore B = 90°$.

(2) When $b \sin A < a$, $\sin B$ has a given value less than unity; but there are two angles less than 180° which have a given sine, one being the supplement* of the other: so there will be two values of B which will satisfy the given conditions. On this account IV. is called the **ambiguous case** in the solution of triangles.

When we find two values of B we shall of course find two of C, and then two of c from the formula $c = \frac{a \sin C}{\sin A}$.

142. We can also make this clear geometrically.



Let X A C be the given angle, A C a given side b. From C draw C N perpendicular to A X.

Then $CN = b \sin A$.

Let a be the other given side, then—

(α) **If a is less than C N** ($=$ C M) a circle described with C M as radius will not meet A X; therefore the triangle will be impossible if $a < CN$, which is $b \sin A$.

(β) **If a is equal to C N,** a circle described with C N as radius, will just touch A X at N. Therefore if $a = b \sin A$,

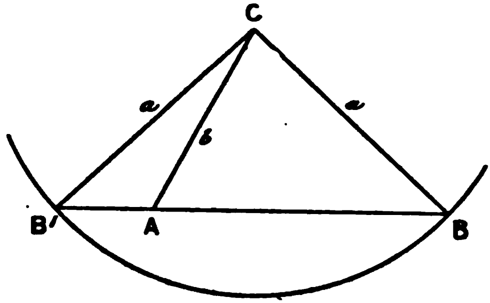
* See page 104, § 87.

only one triangle will be possible, which will have a right angle at N.

(γ) **If a is greater than C N but less than b,** a circle described with a as radius, will cut A X in two points, B and B′. We therefore have two triangles, C A B and C A B′, each having two sides equal to a and b respectively, and the angle opposite to a equal to A.

The angle A B C = 180° − C B B′, = 180° − C B′ A.

That is, the two values of B are supplementary.



(δ) **If a is greater than b** there is only one triangle, because in the triangle C A B′, although C B′ = a, the angle opposite it is 180° − A.

143. *Note.*—The whole subject may be treated as follows:—

In any triangle $a^2 = b^2 + c^2 - 2\,b\,c\cos A$; if a, b, and A are known, this becomes a quadratic in c, namely,

$$c^2 - 2\,b\,c\cos A + b^2 - a^2 = 0.$$

Therefore from the Theory of Quadratics there will be two real values of c if

$$b^2\cos^2 A > b^2 - a^2, \quad \text{or,} \quad a^2 > b^2\sin^2 A,$$
$$\text{or,} \quad a > b\sin A;$$

two equal values of c if $\quad a = b\sin A$;

no real values of c if $\quad a < b\sin A$.

If c_1 and c_2 be the two real values, when there is ambiguity, we have at once from the Theory of Quadratic Equations—

$$c_1 + c_2 = 2\,b\cos A \quad . \quad . \quad . \quad . \quad (1)$$
$$c_1\,c_2 = b^2 - a^2 \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Example.—In the ambiguous case, if c_1 and c_2 be the values of the third side, and $c_1^2 + c_1 c_2 + c_2^2 = a^2$, show that $A = 60°$ or $120°$.

$$\begin{array}{lrl} \text{From (1)} & c_1^2 + 2c_1 c_2 + c_2^2 & = 4b^2 \cos^2 A \\ \text{and we have given} & c_1^2 + c_1 c_2 + c_2^2 & = a^2 \\ \hline \text{subtracting} & c_1 c_2 & = 4b^2 \cos^2 A - a^2. \\ \text{Therefore from (2)} & b^2 - a^2 & = 4b^2 \cos^2 A - a^2, \end{array}$$

$$\therefore \cos^2 A = \frac{1}{4}, \quad \therefore \cos A = \pm\frac{1}{2},$$

$$\text{or, } A = 60° \text{ or } 120°.$$

EXAMPLES XXIII (*e*).

[Where there is ambiguity always give both solutions in full.]

(1) $a = 2$, $b = \sqrt{6}$, $A = 45°$; solve the triangle.

(2) $b = 3$, $c = 3\sqrt{2}$, $B = 30°$; solve the triangle.

(3) $b = 145$, $a = 178$, $A = 41° 10'$; find B and C.

Given

$\log 178 = 2 \cdot 2511513$, $\quad L \sin 41° 10' = 9 \cdot 8183919$,

$\log 145 = 2 \cdot 1613680$, $\quad L \sin 32° 21' 54'' = 9 \cdot 7286086$.

(4) $a = 8$, $b = 7$, $A = 120°$; find B and C.

Given $\log 2 = \cdot 3010300$, $\quad L \sin 49° 16' = 9 \cdot 8795287$,

$\log 3 = \cdot 4771213$, $\quad L \sin 49° 17' = 9 \cdot 8796375$,

$\log 7 = \cdot 8450980$.

(5) $c = 320$, $a = 468$, $C = 32° 15'$; find B.

Given $\log 2$ and $\log 3$, $\quad L \sin 32° 15' = 9 \cdot 7272276$,

$\log 13 = 1 \cdot 1139434$, $\quad L \sin 51° 18' = 9 \cdot 8923236$.

(6) If in the preceding question the values of c and a be interchanged, find A and B; given in addition

$$L \sin 21° 23' = 9 \cdot 5621316.$$

(7) $a = 250$, $b = 240$, $A = 72° 4' 48''$; find the other angles.

$\log 2\cdot5 = \cdot3979400$, $\quad L \sin 72° 4' = 9\cdot9783702$,

$\log 2\cdot4 = \cdot3802112$, $\quad L \sin 72° 5' = 9\cdot9784111$,

$L \sin 65° 59' = 9\cdot9606739$.

(8) Show that the difference between the two values of c, when a, b, and A are given, is $2\sqrt{a^2 - b^2 \sin^2 A}$.

(9) If the given angle A be 45° and c_1 and c_2, the two values of the ambiguous side, show that the cosine of the angle between the two positions of a is $\dfrac{2\, c_1\, c_2}{c_1{}^2 + c_2{}^2}$.

(10) Given a, b, and A, if c_1, c_2, be the two values of the third side, and B_1, B_2, the angles opposite the side b, show that $(c_1 - c_2) \cot A = c_1 \cot B_1 - c_2 \cot B_2$.

CHAPTER XXIV.

MEASUREMENT OF HEIGHTS AND DISTANCES.

144. In Chapter VII. we gave some easy examples on practical measurements. The principles established in the later chapters enable us to solve more complicated problems, and the calculations required can often be simplified by the help of logarithms.

As before, no definite rules for solution can be laid down, but we shall divide the subject into three sections, giving a characteristic example in each case.

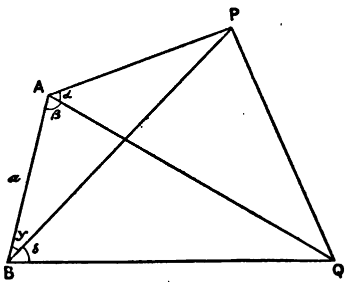
A. **Measurements all in the same plane.**

145. The calculations can often be made from right-angled triangles; if not, the formula which we should always try to apply is,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C},$$

as it is simple and easily expressed in logarithmic form.

Example.—Find the distance of each of two inaccessible forts from a certain point.



Let A be the given point, P and Q the forts. Measure off a distance A B = a in any direction in the same plane as A, P,

and Q. Then measure the angles P A Q (a), B A Q (β), at A, and A B P (γ), P B Q (δ), at B.

Then in the triangle P A B we have,

$$\frac{\text{A P}}{\sin \text{A B P}} = \frac{\text{A B}}{\sin \text{A P B}},$$

or

$$\frac{\text{A P}}{\sin \gamma} = \frac{a}{\sin (180^\circ - a - \beta - \gamma)}.$$

$$\therefore \quad \text{A P} = \frac{a \sin \gamma}{\sin (a + \beta + \gamma)}.$$

Similarly, in the triangle A B Q,

$$\frac{\text{A Q}}{\sin \text{A B Q}} = \frac{\text{A B}}{\sin \text{A Q B}},$$

or

$$\frac{\text{A Q}}{\sin (\gamma + \delta)} = \frac{a}{\sin (180^\circ - \beta - \gamma - \delta)}.$$

$$\therefore \quad \text{A Q} = \frac{a \sin (\gamma + \delta)}{\sin (\beta + \gamma + \delta)}.$$

Suppose we have measured A B to be 200 yards, and the angles

$$a = 42^\circ\ 27',\ \beta = 58^\circ\ 13',\ \gamma = 32^\circ\ 19',\ \delta = 43^\circ\ 54';$$

then $\log \text{A P} = \log 200 + \text{L} \sin 32^\circ\ 19' - \text{L} \sin 47^\circ\ 1'$

[sin 132° 59′ = sin 47° 1′]

$= 2{\cdot}3010300 + 9{\cdot}7280275 - 9{\cdot}8642452$ (from tables)

$= 2{\cdot}1648123.$

Whence **A P = 146·16 yards** (from tables).

Also $\log \text{A Q} = \log 200 + \text{L} \sin 76^\circ\ 13' - \text{L} \sin 45^\circ\ 34'$

$= 2{\cdot}30103 + 9{\cdot}9873103 - 9{\cdot}8537381$ (from tables)

$= 2{\cdot}4346022.$

Whence **A Q = 272·02 yards.**

EXAMPLES XXIV (*a*).

(1) A man walking towards a church observes the nave to be $30°$ in elevation, and the spire behind it $45°$. After walking 75 feet further, he sees the nave hide the spire at an elevation of $60°$. Find the height of both nave and spire.

(2) From the end of a pier, 300 yards long and at right angles to the shore, a vessel is observed in direction E.S.E., and from the shore end of the pier it lies in direction S.E. The shore runs E.N.E and W.S.W. Find the distance of the vessel from the shore.

(3) The topmost stone of a tower 201 feet high is one foot thick. Show that it subtends an angle of $6'$ at a place on the ground 100 feet from the foot of the tower; having given

$$\tan 63° \, 27' = 2, \qquad \tan 63° \, 33' = 2 \cdot 01.$$

(4) A man who is walking on a level plain towards a tower observes at a certain point that the elevation of the top of the tower is $10°$, and, after going 50 yards nearer to the tower, that the elevation is $15°$.

Having given

$$L \sin 15° = 9 \cdot 4129962, \qquad \log 25 \cdot 783 = 1 \cdot 4113334,$$
$$L \cos 5° = 9 \cdot 9983442, \qquad \log 25 \cdot 784 = 1 \cdot 4113503,$$

find the height of the tower in yards to four places of decimals.

(5) Two objects, A and B, were observed from a ship to be at the same instant in a line with a bearing N. $15°$ E. The ship then sailed N.W. for 5 miles, when it was observed that A bore E., and B N.E. Find the distance between A and B.

(6) At a point a feet from the foot of a tower of height h standing on a horizontal plane, the angle subtended by the tower is equal to that subtended by its spire. Show that the height of the spire is $\dfrac{a^2 + h^2}{a^2 - h^2} \cdot h$.

(7) A person standing on the bank of a river observes that the top of a tower on the edge of the opposite side subtends an angle of $55°$ with a horizontal line drawn through his eye. Receding backwards 30 feet, he then finds it to subtend an angle of $48°$. Find the breadth of the river.

$$L \sin 7° = 9 \cdot 08589, \qquad \log 3 = \cdot 47712,$$
$$L \sin 35° = 9 \cdot 75859, \qquad \log 1 \cdot 0493 = \cdot 02089.$$
$$L \sin 48° = 9 \cdot 87107,$$

(8) A person at a distance a from a tower, which stands on a horizontal plane, observes that the angle of elevation (α) of its highest point is the complement of that of the top of a flagstaff on the top of it. Show that the length of the flagstaff is $2\,a \cot 2\,\alpha$.

(9) The top mast of a yacht is seen from a point on the deck to subtend the same angle A that the part of the mast below it does. Shew that if the top mast be a feet high, the part below it is $a \cos 2\,A$.

(10) A tower stands on a horizontal plane, and two observers at a known distance (c) apart on either side of it, observe its elevations to be α and β. Shew that the height of the tower is $c \sin \alpha \sin \beta \operatorname{cosec} (\alpha + \beta)$.

(11) In the previous question, if $c = 100$ yards, $\alpha = 46° 51'$, $\beta = 57° 34'$, find h, having given

$$L \sin 46° 51' = 9 \cdot 8630644, \quad L \sin 57° 34' = 9 \cdot 9263507,$$
$$L \sec 14° 25' = 10 \cdot 0138955,$$
$$\log 6357 = 3 \cdot 8032522, \quad \log 6358 = 3 \cdot 8033205.$$

(12) A man walking along a straight road running in a direction 30° east of north, observes that he is due south of a certain house. A mile farther on he is due east of it, and a church on the opposite side of the road is N.E. of him. Three miles farther on he is due north of the church. Find the distance between the house and the church, and the angle which the line joining them makes with the road. Given $\sin 23° 8' = \cdot 3928$.

(13) A man standing between two towers, 200 feet from the base of the higher, which is 90 feet high, observes their altitudes to be the same. He walks 70 feet nearer to the shorter tower, and now he finds that the altitude of one is double that of the other. Find the actual height of the second tower, and his original distance from it.

(14) The captain of a ship observes a lighthouse in a direction making an angle of 47° 30′ with that in which he is sailing; continuing in the same course for a mile he then observes the angle to be 55°. Determine at what distance he will pass the lighthouse.

Given

$$L \sin 55° = 9 \cdot 9133645, \quad L \sin 7\tfrac{1}{2}° = 9 \cdot 1156977,$$
$$L \sin 47\tfrac{1}{2}° = 9 \cdot 8676309, \quad \log 4 \cdot 627 = \cdot 6652977.$$

(15) A man climbs a mountain. From A to B the slope is 30°, from B to C 15°, from C to the summit 75°. He measures

A B = 2000 feet, B C = 2500 feet. The rest is too rough to be measured, but he sees from the summit A and B in one straight line. Find the height of the mountain correct to a foot.

(16) D E is a pillar on a horizontal plane. A B C D is a straight line in the plane. The height of the pillar subtends an angle θ at A, $2\,\theta$ at B, and $3\,\theta$ at C. If A B = 50 feet, and B C = 20 feet, find the height of the pillar, and the distance C D.

(17) A B is a horizontal line of length 400 yards; from a point in A B a balloon ascends vertically, and after a certain time its altitude is taken simultaneously at A and B. At A it is 64° 15′, and at B 48° 20′. Find the height of the balloon when observations are taken.

Given

$$\log 2 = \cdot 3010300, \qquad L \sin 64^\circ\, 15' = 9\cdot 9545793,$$
$$\log 2\cdot 29149 = \cdot 4646213, \qquad L \sin 48^\circ\, 20' = 9\cdot 8733352,$$
$$L \sin 67^\circ\, 25' = 9\cdot 9653532.$$

(18) The sides of a rectangle A B C D are a and b (A B $= a$). A person sees C, D in a line with each other, and A B, A D subtending the same angle. He then walks in a straight line till he sees A, C in a line, and he then finds himself at a distance c from C. Find the angle that his direction of walking makes with C D, and show that it is a right angle if $a\,c = \sqrt{(a^4 - b^4)}$.

(19) A lighthouse was observed by a ship at sea to bear S.E.; after the ship had sailed N.E. for six miles the lighthouse was seen to bear 27° 30′ E. of S.; find the distance of the lighthouse at each moment of observation.

$$\log 2 = \cdot 30103, \qquad L \sin 17^\circ\, 30' = 9\cdot 4781418,$$
$$\log 3 = \cdot 4771213, \qquad L \tan 17^\circ\, 30' = 9\cdot 4987223,$$
$$\log 19\cdot 953 = 1\cdot 3000095,$$
$$\log 19\cdot 029 = 1\cdot 2794160,$$
$$\log 19\cdot 030 = 1\cdot 2794388.$$

(20) To find the length of an inaccessible wall an observer placed himself due south of one end of the wall, and then due west of the other end, in such positions that the angle θ which the wall subtended at the two positions was the same. The distance apart of the two stations being a, shew that the length of the wall was $a \tan \theta$.

(21) A hill slopes at an angle θ to the horizon; on the top of it stands a tower, from the foot of which an observer walks a

distance ($= a$ yards), and the tower subtends at his eye an angle (α); he then proceeds b yards further and the angle subtended by the tower is β. Prove that

$$\frac{\cos(\alpha + \theta)}{a \sin \alpha} = \frac{\cos(\beta + \theta)}{(a + b) \sin \beta}.$$

(22) A B, C D are two towers in the same horizontal plane. The height of A B is h, the elevation of D at A is α, at B is β. Show that the height of C D is $\dfrac{h \sin \alpha \cos \beta}{\sin (\alpha - \beta)}$.

(23) In the last question, if $h = 80$, $\alpha = 30°$, $\beta = 20° \, 15'$, find C D.

Given $\log 2 = \cdot 30103$, $\log 221 \cdot 59 = 2 \cdot 3455502$,

$L \cos 20° \, 15' = 9 \cdot 9722914$, $\log 221 \cdot 6 = 2 \cdot 3455798$,

$L \sin 9° \, 45' = 9 \cdot 2287839$.

(24) A ship sailing due south passes two lighthouses, and a man on board observes that the greatest angle subtended at his eye by the lights is 60°. Half-an-hour afterwards the lights appear both N.W. of the ship. Find the rate at which the ship is sailing, it being known that the lights are at a distance $2 \sqrt{3}$ miles from each other.

(25) A round tower stands on an island in a lake. A B are two points on the land such that A B is a feet, and points directly to the middle of the tower. At A and B the base of the tower subtends angles 2α and 2β respectively. Prove that the diameter of the tower is $\dfrac{2 a \sin \alpha \sin \beta}{\sin \beta - \sin \alpha}$.

(26) A straight flagstaff, leaning due east, is found to subtend an angle α at a point, in the plain on which it stands, a yards west of the base. At a point b yards east of the base the flagstaff subtends an angle β. Find at what angle it leans.

(27) From each of two ships a mile apart the angle is observed which is subtended by the other ship and a beacon on shore; these angles are found to be $52° \, 25' \, 15''$ and $75° \, 9' \, 30''$ respectively.

Having given

$L \sin 75° \, 9' \, 30'' = 9 \cdot 9852635$, $\log 1 \cdot 2197 = \cdot 0862530$,

$L \sin 52° \, 25' \, 15'' = 9 \cdot 8990055$, $\log 1 \cdot 2198 = \cdot 0862886$,

find the distances of the beacon from each of the ships, one distance exact, and the other to five places of decimals.

(28) A man observes the angles of elevation of the top of a tower to be α and β respectively, as seen from two points A and B, situated in a line with the foot of the tower, and at a distance c from each other. He notices also that a flagstaff on the tower subtends equal angles at A and B. Find the height of the flagstaff.

(29) A person notices two objects in a straight line due east. After walking a distance a due north, he observes that the objects subtend an angle α at his eye; and after walking a further distance a, an angle β. Prove that the distance between the objects is $\dfrac{3\,a \tan\alpha \tan\beta}{2\tan\alpha - \tan\beta}$.

(30) A tower stands on a mound, which is inclined at an angle of 15° to the horizon. An observer starts from A on the level plain, where the angle of elevation of the top of the tower is 30°, and walks a distance a to B, the foot of the mound; and then a distance b straight up the mound to C, where the angle of elevation above the slope of the mound is 45°. Find the height of the tower above the horizontal plane AB; and if $a = b\sqrt{2}$, show that C is half-way up the mound, and find the angle of elevation at B.

(31) The elevations of two mountains in the same line with the observer are 9° 30′ and 18° 20′; on his approaching 4 miles nearer they have both an elevation of 37°. Find the heights of the mountains in yards.

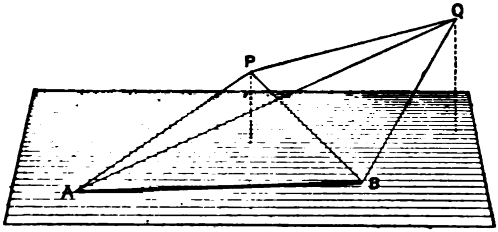
Given $\log 2 = \cdot 30103$,
$\log 11 = 1\cdot 0413927$,
$\log 1420 = 3\cdot 1522883$,
$\log 1420\cdot 1 = 3\cdot 1523189$,
$\log 4163\cdot 6 = 3\cdot 6194690$,
$\log 4163\cdot 7 = 3\cdot 6194794$,
$\log 1594\cdot 397 = 3\cdot 1802393$.

$L \sin 9° \, 30' = 9\cdot 2176092$,
$L \sin 18° \, 20' = 9\cdot 4976824$,
$L \sin 18° \, 40' = 9\cdot 5052339$,
$L \sin 27° \, 30' = 9\cdot 6644056$,
$L \sin 37° = 9\cdot 7794630$,

B. Measurements in different planes.

146. The only additional difficulty in this case will be in drawing the figure so as to get a clear view of the problems involved.

Example.—To find the heights above a given point of two inaccessible mountains, and the distance between their summits.



Let P and Q be the two summits, A the given point.

Measure off a distance A B = c. Measure also the following angles:—

P A B, Q A B, P A Q, at A,

(noticing that P A B is not equal to Q A B + P A Q)

P B A, Q B A at B.

Then (1) In the triangle P A B

Two angles P A B, P B A, and the adjacent side A B are known;

$\therefore$ **we can find A P.**

(2) In the triangle Q A B

Two angles Q A B, Q B A, and the side A B are known;

$\therefore$ **we can find A Q.**

(3) In the triangle P A Q

P A, Q A, and the included angle P A Q are known;

$\therefore$ **we can find P Q.**

(4) Measuring the elevations of P and Q at A we have, if α and β be the angles,

Height of P = A P sin α. Height of Q = A Q sin β.

EXAMPLES XXIV (*b*).

(1) The angular elevation of a tower at a place A due south of it is 30°; at a place B, due west of A, and at a distance a from it, the elevation is 18°. Show that the height of the tower is

$$\frac{a}{\sqrt{2\sqrt{5}+2}}.$$

(2) A regular pyramid stands on a square base; if b is the length of a side of the base, and α the angle this side makes with one of the edges, show that the height of the pyramid is

$$\frac{b}{2\cos\alpha}\cdot\sqrt{-\cos 2\alpha}.$$

(3) Find the height in the last question if $\alpha = 67°$ and $b = 173 \cdot 6$ ft.

Given $\log 2 = \cdot 3010300$

$\log 173\cdot6 = 2\cdot2395497$, $\quad$ $L\cos 67° = 9\cdot5918780$,

$\log 1851516 = 6\cdot2675274$, $\quad$ $L\cos 46° = 9\cdot8417713$.

(4) If A D be drawn perpendicular to the plane of A B C, and D B, D C be joined, show that

$$\sin DCB \,.\, \sin DBA = \sin DCA \,.\, \sin DBC.$$

(5) Being on a river, and observing a column on the bank, I find the angle of elevation of its top to be C, and the angle subtended by its top and a small island down the river to be A. After sailing past the column to this island, a distance of a yards, I find the angle subtended by the top and my former position to be B. Show that the height of the column is $\dfrac{a\sin B\sin C}{\sin(A+B)}$.

(6) In the last question if $a = 480$ yds, $A = 47° 25'$, $B = 18° 30'$, $C = 30°$, find the height.

Given

$\log 2 = \cdot3010300$, $\quad$ $L\sin 18° 30' = 9\cdot5014764$,

$\log 3 = \cdot4771213$, $\quad$ $L\sin 65° 55' = 9\cdot9604484$,

$\log 8\cdot3414 = \cdot9212393$.

(7) The angular elevation of a tower at a place due south of it is 45°, and at another place 200 yards due west of the former

the elevation is 15°. Show that the height of the tower is $300\,(3^{\frac{1}{2}} - 3^{-\frac{1}{2}})$ feet.

(8) A person observes the angular elevation of a balloon due east of him to be 45°. He then travels N.E. while the balloon moves south horizontally until he sees it due south of him, when its angular elevation is $22\frac{1}{2}°$. Show that he and the balloon have passed over equal distances.

(9) The angle of elevation of a tower, which stands due east of an observer is a. Another observer, who is due south of the first, finds that the bearing of the tower is $\beta°$ east of north. Show that the secant of the angle which the distance between the observers subtends at the top of the tower is

$$\sqrt{\sin^2 a + \cos^2 a \operatorname{cosec}^2 \beta}.$$

(10) A man walking along a straight road observes that a certain point to the right of the road has an elevation of 30°, 500 yards further on it has an elevation of 45°, and 300 yards beyond that an elevation of 60°. Find its height.

(11) The angle of elevation of a balloon from a station due south of it is C; from another station due west of the former its elevation is D. If a is the distance between the stations, and x the height of the balloon, show that

$$x^2 = \frac{a^2}{\cot^2 D - \cot^2 C}.$$

(12) In the last question if $a = 1$ mile, $C = 59° 14'$, $D = 43° 37'$, find x in yards.

Given

$$\log 1760 = 3\cdot2455127, \qquad \log 2036\cdot33 = 3\cdot3088484,$$

$$L \sin 59° 14' = 9\cdot9341234, \qquad L \sin 102° 51' = 9\cdot9889849,$$

$$L \sin 43° 37' = 9\cdot8387422, \qquad L \sin 15° 37' = 9\cdot4300750.$$

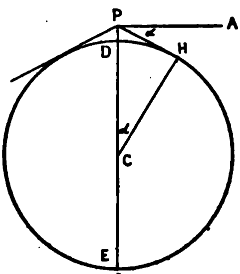
(13) A tree growing vertically on the side of a hill which rises due north at an inclination of 30°, had the upper part broken at 12 feet from the ground by a S.W. wind. If the part fell so that the top touched the ground at 40 feet from the bottom, show that the original height of the tree in feet is

$$12 + \sqrt{1744 - \frac{960}{\sqrt{7}}}.$$

C. Measurements which involve the "Dip of the Horizon."

147. Since the surface of the earth is not plane but spherical, it is obvious that an object on it will only be visible for a certain distance depending on its height, and conversely that at a certain height above the ground the visible horizon will be limited.

148. The angle of depression of the horizon, a in the figure, measured from a point above the earth's surface, is called the "dip of the horizon."



149. This angle is equal to the angle subtended at the centre of the earth by P the point of observation and H the limit of the horizon.

For A P H = the complement of H P C = P C H.

150. *Example.*—**To find D H or P H.**

(1) Given a, and the earth's radius, we have

$$\frac{DH}{DC} = \text{circular measure of } a;$$

$$\therefore DH \text{ is known.}$$

Thus if $a = 9'$ and the earth's radius be 3990 miles, we have

$$\frac{DH}{DC} = \frac{9 \times \pi}{180 \times 60}.$$

$$DH = \frac{9 \times 22 \times 3990}{180 \times 60 \times 7} = \mathbf{10 \cdot 45 \text{ miles.}}$$

(2) Given the height of P above the surface.

Imagine that a line through P and C meets the circumference at D and E. Then $PH^2 = PD \cdot PE$.

But PE may be taken equal to the diameter of the earth, since PD is practically very small compared with DE;

$$\therefore\ PH^2 = 2rPD.$$

$$\text{If } PD = 220 \text{ ft., } r = 3990,$$

$$PH^2 = \frac{2 \times 3990 \times 220}{1760 \times 3} \text{ in miles} = 332 \cdot 5.$$

$$\therefore\ \mathbf{PH = 18 \cdot 23 \text{ miles.}}$$

EXAMPLES XXIV (c).

(1) Two hills 264 ft. high are just visible from each other over the sea; how far are they apart?

(take the radius of the earth 4000 miles.)

(2) If two points eight miles apart each ten feet above the water were just visible from each other; what would be the earth's radius?

(3) If the earth's radius is 3990 miles at the place, what distance apart must the two points in the last question be so as to be just visible to each other?

Given $\log 1995 = 3 \cdot 2999429$, $\log 3 \cdot 8876 = \cdot 5896816$,
$\log 132 = 2 \cdot 1205739$, $\log 3 \cdot 8877 = \cdot 5896927$.

(4) From the top of a cliff the angle of depression of the horizon is found to be 10′. Show that the height of the cliff is $\frac{1}{2} \cdot \frac{\pi^2 r}{1080^2}$ miles, where r is the earth's radius.

(5) In the last question, taking $\pi = 3 \cdot 14159$, $r = 3990$, find the height in yards.

Given $\log 3 \cdot 1415 = \cdot 4971371$, $\log 108 = 2 \cdot 0334238$,
$\log 3 \cdot 1416 = \cdot 4971509$, $\log 1760 = 3 \cdot 2455127$,
$\log 1995 = 3 \cdot 2999429$, $\log 29 \cdot 71 = 1 \cdot 4729072$.

(6) From the mast of a ship the top of a lighthouse known to be 500 ft. above the sea is just visible at a depression of 9′ 27″. How far is the ship from the lighthouse?

Given $\log 132 = 2 \cdot 1793690$, $\log 25 \cdot 7226 = 1 \cdot 4103155$.

$$\text{Take } \pi = \frac{22}{7},\ r = 4000$$

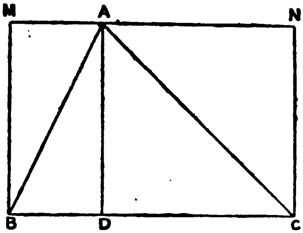
CHAPTER XXV.

ON THE AREA OF A TRIANGLE, AND THE RADII OF CERTAIN CIRCLES CONNECTED WITH IT.

A. On the area of a triangle (denoted always by S or Δ).

151. The area of a triangle is equal to half the product of two of the sides into the sine of the included angle.

(1) Let A B C be an acute angled triangle. Draw A D perpendicular to the base B C. Through A draw M N parallel to B C, and through B and C, B M and C N parallel to A D. Then B M N C is a rectangle having the same base B C as the triangle A B C and lying between the same parallels.



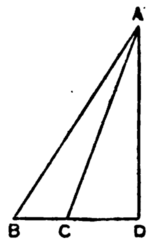
$$\therefore\ S = \frac{1}{2} BMNC = \frac{1}{2} BC \cdot BM = \frac{1}{2} AD \cdot BC;$$

but
$$\frac{AD}{AC} = \sin C;$$

$$\therefore\ S = \frac{1}{2} AC \cdot BC \sin C = \frac{1}{2} ab \sin C.$$

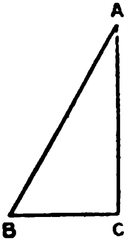
(2) When C is an obtuse angle,

as before,



$$S = \frac{1}{2} BC \cdot AD = \frac{1}{2} BC \cdot AC \sin ACD$$

$$= \frac{1}{2} ab \sin (180^\circ - C) = \frac{1}{2} ab \sin C.$$



(3) When C is a right angle,

$$S = \frac{1}{2} BC \cdot AC = \frac{1}{2} ab$$

$$= \frac{1}{2} ab \sin C, \text{ since}$$

$$\sin C = \sin 90^\circ = 1.$$

In the same way S may be shown equal to $\frac{1}{2} ca \sin B$, or $\frac{1}{2} bc \sin A$.

152. $\mathbf{S = \sqrt{s(s-a)(s-b)(s-c)}}$

for $S = \frac{1}{2} bc \sin A$

$$= \frac{1}{2} bc \times \frac{2}{bc} \sqrt{s(s-a)(s-b)(s-c)}$$

$$= \sqrt{s(s-a)(s-b)(s-c)}.$$

Note.—When multiplied out

$$s(s-a)(s-b)(s-c) = \frac{1}{16}(2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4),$$

a form which is sometimes used.

EXAMPLES XXV (*a*).

Find the area of the triangle in the following cases:—

(1) $a = 4$ ft., $b = 6$ ft., $C = 30^\circ$.

(2) $b = 7$ ft., $c = 5\sqrt{2}$ ft., $A = 135^\circ$.

(3) $a = 2(\sqrt{3} + 1)$ inches, $B = 45^\circ$, $C = 60^\circ$.

(4) $a = \cdot 9$, $b = 1\cdot 2$, $c = 1\cdot 5$ inch.

(5) $A = 30^\circ$, $a = 100$, $c = 100\sqrt{3}$ (two solutions).

(6) $a = 7\cdot 152$ in., $b = 8\cdot 263$ in., $c = 9\cdot 375$ in.

Given $\log 1\cdot 2395 = \cdot 0932465$, $\log 3\cdot 02 = \cdot 4800069$,
$\log 5\cdot 243 = \cdot 7195799$, $\log 2\cdot 8477 = \cdot 4544942$,
$\log 4\cdot 132 = \cdot 6161603$, diff. $= 152$.

Show that the value of S can be expressed in the four following forms:—

(7) $$\frac{c \sin A \sin B}{a \sin A + b \sin B + c \sin C} \times \frac{a^2 + b^2 + c^2}{2}.$$

(8) $$\frac{a^2 + b^2 + c^2}{4 (\cot A + \cot B + \cot C)}.$$

(9) $$\frac{2\,a\,b\,c}{a + b + c} \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}.$$

(10) $$\sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \left(\frac{a^2}{\sin A} + \frac{b^2}{\sin B} + \frac{c^2}{\sin C}\right).$$

(11) The sides of a triangle are in A. P., common difference 2 inches. If the area be $3\sqrt{15}$ square inches, find the sides.

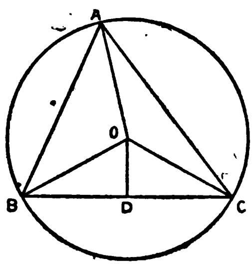
(12) In the three edges of a cube which meet at one angle three points A, B, C are taken, at distances a, b, c, respectively, from that angle. Prove that the area of A B C is

$$\frac{1}{2} \sqrt{b^2 c^2 + c^2 a^2 + a^2 b^2}.$$

B. On the length of the Radius of the Circumscribed Circle (denoted by R).

153. To prove

$$R = \frac{a}{2 \sin A} = \frac{b}{2 \sin B} = \frac{c}{2 \sin C}.$$



Let A B C be the triangle, O the centre of the circle described about it.

Then $OA = OB = OC = R$.

Draw O D perpendicular to B C.

Then $BD = DC = \frac{a}{2}$.

and $\angle BOD = \angle DOC$. [Euc. iv. 5.]

But the angle $BOC = 2A$. [Euc. iii. 20.]

$\therefore \angle BOD = A$;

so, $$\sin A = \sin BOD = \frac{BD}{BO} = \frac{a}{2R}.$$

$$\therefore R = \frac{a}{2 \sin A} = \frac{b}{2 \sin B} = \frac{c}{2 \sin C}.$$

o

154. To prove $R = \dfrac{abc}{4S}$.

This follows at once from the above, and from § 151, for

$$R = \frac{a}{2\sin A} = \frac{abc}{2bc\sin A} = \frac{abc}{4S}.$$

EXAMPLES XXV (*b*).

Find R in the following cases :—

(1) $a = 3$ in., $b = 4$ in., $c = 5$ in.

(2) $b = 4$ ft., $A = 27°$, $C = 93°$.

(3) $a = 25$ in., $b = 39$ in., $c = 40$ in.

(4) $a = 52\cdot375$ in., $A = 27° \, 34' \, 15''$.

Given

$\log 2\cdot6187 = \cdot4180857$,	$L \sin 27° \, 34' = 9\cdot6653749$,
$\log 2\cdot6188 = \cdot4181023$,	$L \sin 27° \, 35' = 9\cdot6656168$.
$\log 56\cdot579 = 1\cdot7526553$,	
$\log 56\cdot580 = 1\cdot7526629$,	

Prove that the four following expressions are all values of R.

(5) $\dfrac{a\cos A + b\cos B + c\cos C}{4\sin A\sin B\sin C}$.

(6) $\sqrt{\dfrac{bc}{2\{\cos(B - C) + \cos A\}}}$.

(7) $\dfrac{1}{8}(a + b + c)\sec\dfrac{A}{2}\sec\dfrac{B}{2}\sec\dfrac{C}{2}$.

(8) $\sqrt{\dfrac{a^2 + b^2 + c^2}{8(1 + \cos A\cos B\cos C)}}$.

(9) If a', b', c' be the perpendiculars from the centre of the circumscribed circle on the sides a, b, c respectively, then

$$\frac{a}{a'} + \frac{b}{b'} + \frac{c}{c'} = \frac{1}{4}\cdot\frac{abc}{a'b'c'}.$$

(10) If O be the centre of the circumscribing circle, and R′ be the radius of the circle described round the triangle B O C, prove that $2R'\cos A = R$.

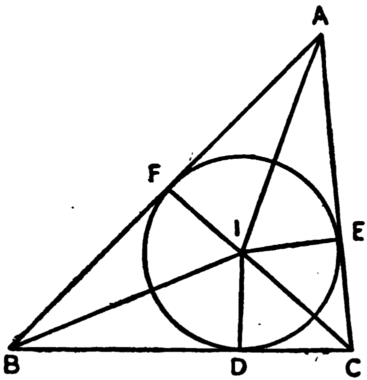
(11) A D is the diameter of the circumscribing circle, and D F the tangent at D meets A C produced in F. Show that the area of D C F is $\frac{1}{2} b^2 \cot^3 B$.

(12) The bisector of the angle A meets B C in D, and the circumscribed circle in E; show that

$$DE = \frac{a \tan \frac{A}{2}}{2 \cos \frac{B-C}{2}}.$$

C. **On the length of the radius of the circle inscribed in a triangle** (denoted by r).

155. Let A B C be the triangle, I the centre of the inscribed circle. Draw I D, I E, I F perpendicular to B C, C A, A B respectively, then $ID = IE = IF = r$.



Join I A, I B, I C.

Now the triangle A B C is equal to the sum of the triangles I B C, I A C, I A B;

or, $$S = \frac{1}{2} . BC . ID + \frac{1}{2} CA . IE + \frac{1}{2} AB . IF,$$

or, $$S = \frac{1}{2} r (a + b + c) = r s.$$

$\therefore$ $$r = \frac{S}{s}.$$

This may be put into another shape;

$$r = \frac{S}{s} = \frac{\sqrt{s(s-a)(s-b)(s-c)}}{s} = (s-a)\frac{\sqrt{(s-b)(s-c)}}{\sqrt{s(s-a)}}$$

$$\therefore\ \ \mathbf{r = (s-a)\tan\frac{A}{2}}.$$

This last result is better proved independently, thus—

Since

$$AF = AE;\ BF = BD;\ CE = CD;\ \text{(by geometry)},$$

we have

$$AF + BD + CD = \frac{1}{2}(AB + BC + CA),$$

that is

$$AF + a = \frac{1}{2}(a + b + c) = s.$$

$$\therefore\ \ AF = s - a.$$

So

$$r = IF = AF \tan IAF = (s-a)\tan\frac{A}{2};$$

similarly we prove r equal to $(s-b)\tan\frac{B}{2}$ and $(s-c)\tan\frac{C}{2}$.

EXAMPLES XXV (*c*).

(1) Find the radius of a circle inscribed in a triangle whose sides are 25, 39 and 40 inches in length.

(2) The sides of a triangle are 3, 7 and 8 ft., show that $R : r :: 7 : 2$.

(3) If the sides are as $3 : 5 : 6$, then $R : r :: 45 : 16$.

Prove the following (I being the centre of the inscribed circle).

(4) $r \cos\frac{A}{2} = a \sin\frac{B}{2}\sin\frac{C}{2}$.

(5) $r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$.

(6) $S = r^2\left\{\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2}\right\}$.

(7) $S = Rr\,\{\sin A + \sin B + \sin C\}$.

(8) $IB\,.\,IC = ar \sec\frac{A}{2}$.

(9) $\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca} = \frac{1}{2Rr}$.

(10) $AI\cos\frac{A}{2} + BI\cos\frac{B}{2} + CI\cos\frac{C}{2}$

$= 4R\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$.

(11) If AD drawn through I meet BC at D, the sum of the radii of circles circumscribing ABD, ACD is equal to $2R\cos\frac{A}{2}$.

(12) If D, E, F are the points of contact of the inscribed circle with a, b, c, then

$$IA\,.\,IB\,.\,IC = 4R\,.\frac{AF\,.\,BD\,.\,CE}{AF + BD + CE}.$$

(13) $(a + b + c)\,r = IA^2\sin A + IB^2\sin B + IC^2\sin C$.

(14) The product of the perpendiculars from the angles of a triangle on the opposite sides is

$$\frac{(a+b+c)^3}{abc}r^3.$$

(15) In a triangle right-angled at C

$$\frac{c}{r} = \frac{c+a}{b} + \frac{c+b}{a}.$$

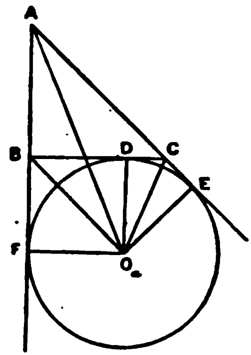
(16) A person close to the circular base of a tower finds that he can see a certain distant object by walking either a distance p in one direction, or q in the opposite direction, along the tangent line to the base. Show that the distance of the object from the centre of the base is $\frac{r\sqrt{(r^2+p^2)(r^2+q^2)}}{pq - r^2}$, r being the radius of the base.

D. On the length of the radius of an Escribed Circle.

156. **Definition.**—A circle which touches **one side** of a triangle and the other **two sides produced** is called an **escribed** circle. There are three such for every triangle, one touching each of the sides a, b, c, and their radii are denoted by r_a, r_b, r_c, respectively.

157. To prove that $r_a = \dfrac{S}{s - a}$.

Let A B C be a triangle, and let the exterior angles at B and C be bisected by straight lines meeting at O_a.



Then if O_aD, O_aE, O_aF be drawn perpendicular to B C and to A B and A C produced, it is easily shown as for the inscribed circle that $O_a D = O_a E = O_a F$, and that, therefore, a circle with centre O_a and any one of these as radius will touch the straight lines B C, A E, A F at D, E, and F.

Now $\quad \Delta\, ABC + \Delta\, BCO_a = \Delta\, ABO_a + \Delta\, ACO_a$

(since each pair is equal to the quadrilateral ABO_aC).

$$\therefore \quad \Delta\, ABC = \Delta\, ABO_a + \Delta\, ACO_a - BCO_a,$$

or,
$$S = \frac{1}{2}\{AB \,.\, O_aF + AC \,.\, O_aE - BC \,.\, O_aD\}$$

$$= \frac{1}{2}\{b\, r_a + c\, r_a - a\, r_a\}$$

$$= r_a \,.\, \frac{b + c - a}{2} = r_a\,(s - a).$$

$$\therefore \quad r_a = \frac{S}{s - a};$$

similarly $\quad r_b = \dfrac{S}{s - b}$ and $r_c = \dfrac{S}{s - c}$.

158. To prove $r_a = s \tan \frac{A}{2}$.

In the equal triangles $A F O_a$, $A E O_a$, we have $A F = A E$;

and $$\angle F A O_a = \angle E A O_a = \frac{A}{2}.$$

Also $$B F = B D \text{ and } C E = C D.$$

$$\therefore \quad A F = \frac{1}{2}(A F + A E) = \frac{1}{2}(A B + B D + A C + C D)$$

$$= \frac{1}{2}(A B + B C + C A) = s,$$

so, $$r_a = O_a F = A F \tan F A O_a = s \tan \frac{A}{2}$$

similarly,

$$r_b = s \tan \frac{B}{2} \text{ and } r_c = s \tan \frac{C}{2}.$$

EXAMPLES XXV (*d*).

Prove the following:—

(1) $R r_a (s - a) = R r_b (s - b) = R r_c (s - c) = \dfrac{a b c}{4}$.

(2) $r_a r_b + r_b r_c + r_c r_a = \left(\dfrac{a + b + c}{2}\right)^2$.

(3) $S = \dfrac{r_a r_b r_c}{\sqrt{r_a r_b + r_b r_c + r_c r_a}}$.

(4) $r_a r_b r_c = a b c \cos \dfrac{A}{2} \cos \dfrac{B}{2} \cos \dfrac{C}{2}$.

(5) $S = \sqrt{r r_a r_b r_c}$.

(6) $\dfrac{2 s}{r} + \dfrac{a}{r_a} + \dfrac{b}{r_b} + \dfrac{c}{r_c} = 8 R\left(\dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c}\right)$.

(7) The distances between the centre of the inscribed circle and the centres of the escribed circles are $a \sec \frac{A}{2}$, $b \sec \frac{B}{2}$, $c \sec \frac{C}{2}$.

(8) The lengths of the sides of a triangle formed by the centres of the three escribed circles are $a \operatorname{cosec} \frac{A}{2}$, $b \operatorname{cosec} \frac{B}{2}$, $c \operatorname{cosec} \frac{C}{2}$.

(9) The area of this triangle is $\frac{1}{2} b c \cos \frac{A}{2} \operatorname{cosec} \frac{B}{2} \operatorname{cosec} \frac{C}{2}$.

(10) If r_a, r_b, r_c are in H . P the sides a, b, c are in A . P.

(11) Show that $r_a + r_b + r_c = r + 4R$.

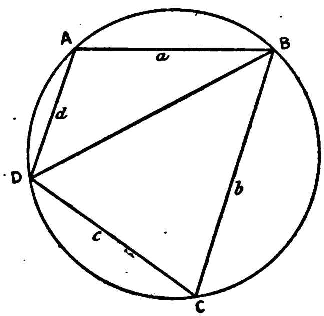
(12) In any triangle show that

$$a b + b c + c a = r^2 + s^2 + 4 R r.$$

CHAPTER XXVI.

CIRCLES AND POLYGONS. THE AREA OF A CIRCLE.

159. To find the area of a quadrilateral inscribed in a circle.



Let A B C D be a quadrilateral inscribed in a circle, a, b, c, d its sides; it is required to find its area Δ. Join D B.

$$\Delta = \text{triangle } ADB + \text{triangle } DBC$$

$$= \frac{1}{2}(ad \sin A + bc \sin C); \text{ but } A + C = 180°.$$

(Euc. III. 22.)

$$\therefore \quad \sin C = \sin(180° - A) = \sin A;$$

$$\therefore \quad \Delta = \frac{1}{2}(ad + bc)\sin A \quad . \quad . \quad . \quad . \quad (1)$$

Now $BD^2 = a^2 + d^2 - 2ad \cos A$ from the triangle A D B

$= b^2 + c^2 - 2bc \cos C$,, ,, D B C

$= b^2 + c^2 + 2bc \cos A$,

since $\cos C = \cos(180° - A) = -\cos A$.

$$\therefore \quad b^2 + c^2 + 2bc \cos A = a^2 + d^2 - 2ad \cos A;$$

$$\therefore \quad \cos A(2bc + 2ad) = a^2 + d^2 - b^2 - c^2;$$

or, $$\cos A = \frac{a^2 + d^2 - b^2 - c^2}{2bc + 2ad}.$$

So
$$1+\cos A=\frac{2bc+2ad+a^2+d^2-b^2-c^2}{2bc+2ad}$$
$$=\frac{(a+d)^2-(b-c)^2}{2bc+2ad}$$
$$=\frac{(a+d+b-c)(a+d-b+c)}{2bc+2ad}$$
$$1-\cos A=\frac{2bc+2ad-a^2-d^2+b^2+c^2}{2bc+2ad}$$
$$=\frac{(b+c)^2-(a-d)^2}{2bc+2ad}$$
$$=\frac{(b+c+a-d)(b+c-a+d)}{2bc+2ad}$$
$$\therefore \sin^2 A=(1+\cos A)(1-\cos A)$$
$$=\frac{(b+c+d-a)(a+c+d-b)(a+b+d-c)(a+b+c-d)}{(2bc+2ad)^2}.$$

Let
$$a+b+c+d=2s,$$
then
$$b+c+d-a=2s-2a, \text{ \&c.}$$
$$\therefore \quad \sin^2 A=\frac{(2s-2a)(2s-2b)(2s-2c)(2s-2d)}{4(bc+ad)^2}$$
$$=\frac{4(s-a)(s-b)(s-c)(s-d)}{(bc+ad)^2};$$
$$\sin A=\frac{2\sqrt{(s-a)(s-b)(s-c)(s-d)}}{bc+ad} \quad . \; . \; . \quad (2)$$

$\therefore \Delta=\sqrt{(\mathrm{s}-\mathrm{a})(\mathrm{s}-\mathrm{b})(\mathrm{s}-\mathrm{c})(\mathrm{s}-\mathrm{d})}$, substituting in (1) from (2).

160. To find the radius of the circle circumscribing a quadrilateral whose sides are given.

Let R denote the given radius; then with the previous notation
$$R=\frac{BD}{2\sin A}. \quad (\S\ 153.)$$

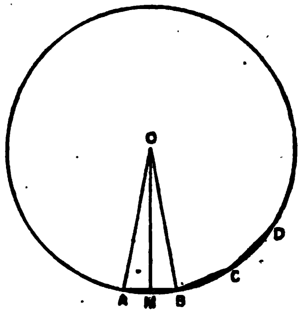
Also
$$BD^2=a^2+d^2-2ad\cos A$$
$$=a^2+d^2-ad\,\frac{(a^2+d^2-b^2-c^2)}{bc+ad}$$
$$=\frac{bc(a^2+d^2)+ad(b^2+c^2)}{bc+ad}$$
$$=\frac{(ac+bd)(ab+cd)}{(bc+ad)}.$$

$$\therefore \quad BD = \frac{\sqrt{(ac+bd)(ab+cd)}}{\sqrt{bc+ad}};$$

$$\therefore \quad R = \frac{\sqrt{(ac+bd)(ab+cd)} \times (bc+ad)}{2\sqrt{bc+ad} \,.\, 2\sqrt{(s-a)(s-b)(s-c)(s-d)}};$$

$$\therefore \quad \mathbf{R} = \frac{\sqrt{(\mathbf{ab+cd})(\mathbf{bc+ad})(\mathbf{ac+bd})}}{4\sqrt{(\mathbf{s-a})(\mathbf{s-b})(\mathbf{s-c})(\mathbf{s-d})}}.$$

161. To find the length of the side and the area of a regular polygon of n sides inscribed in a circle.



Let A B be the side of a regular polygon of n sides A B C D... inscribed in a circle whose centre is O and radius r. Join O A, O B, and draw O M perpendicular to A B. Then the triangles O A M and O M B are equal in every respect (by Eucl. III. 3).

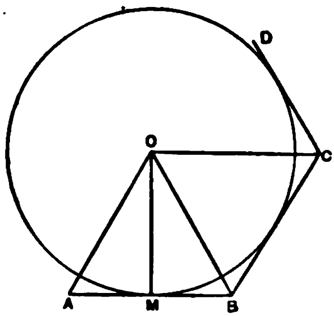
Now the angle A O B is the nth part of four right angles $= \frac{2\pi}{n}$; therefore the angle A O M $= \frac{\pi}{n}$

$$\mathbf{AB} = 2\,AM = 2\,AO \,.\, \sin AOM = \mathbf{2\,r \,.\, \sin\frac{\pi}{n}} \quad (1)$$

The area of the triangle A O B $= \frac{1}{2}$ A O . B O sin A O B $= \frac{1}{2} r^2 \sin \frac{2\pi}{n}$; but the area of the whole polygon is n times that of the triangle A O B,

$$\therefore \ \textbf{Area of polygon} = \frac{\mathbf{n\,r^2}}{\mathbf{2}} \,\mathbf{sin}\, \frac{\mathbf{2\,\pi}}{\mathbf{n}} \quad . \quad . \quad (2)$$

162. To find the length of the side and the area of a regular polygon of n sides circumscribing a circle.



Let A B C D. . . . be the polygon; as before let r be the radius of the circle. Let M be the point of contact of A B; join O M; then M is the middle point of A B, and O M is perpendicular to A B.

The angle

$$A O B = \frac{4 \text{ right angles}}{n} = \frac{2\pi}{n}.$$

$$\therefore \text{ the angle } A O M = \frac{\pi}{n};$$

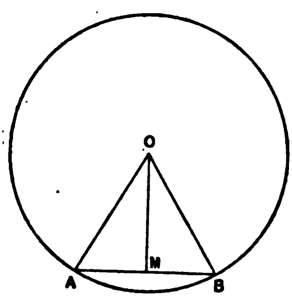
$$\therefore \mathbf{AB = 2\,AM = 2\,OM\,.\,\tan AOM = 2\,r \tan \frac{\pi}{n}} \quad . \; . \; (1)$$

$$\text{The area of the triangle } A O B = \frac{1}{2} A B \, . \, O M$$

$$= r^2 \tan \frac{\pi}{n}.$$

$$\therefore \textbf{ The area of the polygon ABCD} \; . \; . \; . \; . = \mathbf{n\, r^2 \tan \frac{\pi}{n}}.$$

163. To find the area of a circle.



Let O be its centre, A B the side of an inscribed regular polygon of n sides. Draw O M perpendicular to A B.

When n *is gradually increased*, we notice that the differences between—

i. The arc A B and the chord A B,
ii. The sector O A B and the triangle O A B,
iii. The radius O A and the perpendicular O M,

become less and less: therefore, proceeding to the extreme limit, When n is infinitely great, these differences become so small that they may be neglected, and the quantities in i., ii., and iii. respectively, may be considered equal.

The area of the inscribed polygon of n sides $= n$ times the area of the triangle O A B,

$$= n \times \frac{1}{2} \text{A B . O M} \ (\S\ 151),$$

$$= \frac{1}{2} \text{O M} \times \text{perimeter},$$

but when n becomes infinitely great the polygon and the circle circumscribing it become identical, then $\text{O M} = r$, and the perimeter of polygon = circumference of the circle $= 2\pi r$ (§ 8);

$$\therefore \quad \textbf{the area of the circle} = \frac{1}{2} r \times 2\pi r = \pi r^2.$$

EXAMPLES XXVI.

(1) Find the area of a quadrilateral, inscribed in a circle, whose sides are 2, 5, 2, 3 inches.

(2) Also the area of a quadrilateral, and radius of the circle circumscribing it, when the sides are 2, $\sqrt{2}$, 4, and 3 $\sqrt{2}$ inches in length.

(3) Find the perimeter and angle of a regular hexagon inscribed in a circle whose radius is 1 foot.

(4) Also of a regular octagon circumscribing the same circle.

(5) What is the area of the segment of a circle of 3 inches radius cut off by the side of a regular dodecagon inscribed in it?

(6) Show that the square circumscribing a circle is equal in area to $\frac{4}{3}$ of the inscribed regular dodecagon.

(7) Prove that the area of a regular pentagon is to the area of the isosceles triangle used in its construction (Euc. IV. 11) $:: \sqrt{5} : 1$.

(8) The area of the regular hexagon inscribed in a circle is a mean proportional between the areas of the inscribed and circumscribed equilateral triangles.

(9) One circle is inscribed in and another is circumscribed about a regular polygon of n sides. If the length of each side of the polygon is $2a$, find the area of the ring enclosed by the circles.

(10) The side of a regular pentagon inscribed in a circle is 1 inch. Find the radius of the circle to two places of decimals.

(11) Prove that the perimeter of a triangle : the perimeter of an inscribed circle :: the area of the triangle : the area of the circle.

(12) Prove that the areas in the last question are as

$$\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2} : \pi.$$

(13) If a_1, a_2, a_3 be respectively the sides of a regular pentagon, hexagon, and decagon inscribed in a circle, then

$$a_1^2 = a_2^2 + a_3^2.$$

(14) If A be the area of the inscribed circle, A_1 A_2 A_3 the areas of the escribed circles, then $\frac{1}{\sqrt{A}} = \frac{1}{\sqrt{A_1}} + \frac{1}{\sqrt{A_2}} + \frac{1}{\sqrt{A_3}}$.

CHAPTER XXVII.

INVERSE NOTATION.

164. This has been defined in § 38. Before we can prove identities, or solve equations involving quantities expressed in this notation, we must learn to find the sum or difference of two or more angles given in inverse notation, in terms of a third angle similarly expressed. The methods will be best understood by the following illustrations :—

165. To show that $\tan^{-1} a + \tan^{-1} b = \tan^{-1} \dfrac{a+b}{1-ab}$.

This is merely an abbreviated form of the following: Show that the angles whose tangents are a and b are together equal to the angle whose tangent is $\dfrac{a+b}{1-ab}$. The proof requires only a knowledge of the methods of Chapter VIII.

Let $\tan^{-1} a = x$, $\tan^{-1} b = y$; then will $a = \tan x$, $b = \tan y$.

Now $\tan(x+y) = \dfrac{\tan x + \tan y}{1 - \tan x \tan y} = \dfrac{a+b}{1-ab}$;

$$\therefore \quad x + y = \tan^{-1} \frac{a+b}{1-ab};$$

or
$$\tan^{-1} a + \tan^{-1} b = \tan^{-1} \frac{a+b}{1-ab}.$$

Similarly we can show that

$$\tan^{-1} a - \tan^{-1} b = \tan^{-1} \frac{a-b}{1+ab}.$$

**166. To show that $\sin^{-1} a + \sin^{-1} b$
$= \sin^{-1} (a\sqrt{1-b^2} + b\sqrt{1-a^2})$.**

Let $\sin^{-1} a = x$, $\sin^{-1} b = y$; then $a = \sin x$, $b = \sin y$;

$$\therefore \quad \cos x = \sqrt{1 - \sin^2 x} = \sqrt{1-a^2},$$
$$\cos y = \sqrt{1 - \sin^2 y} = \sqrt{1-b^2}.$$

Now $\sin(x+y) = \sin x \cos y + \cos x \sin y$
$$= a\sqrt{1-b^2} + b\sqrt{1-a^2};$$
$$\therefore\ x+y = \sin^{-1}(a\sqrt{1-b^2}+b\sqrt{1-a^2});$$
or $$\sin^{-1}a + \sin^{-1}b = \sin^{-1}(a\sqrt{1-b^2}+b\sqrt{1-a^2}).$$

Similarly we can prove the following:—
$$\sin^{-1}a - \sin^{-1}b = \sin^{-1}(a\sqrt{1-b^2}-b\sqrt{1-a^2})$$
$$\cos^{-1}a + \cos^{-1}b = \cos^{-1}\{ab - \sqrt{(1-a^2)}\sqrt{(1-b^2)}\},$$
$$\cos^{-1}a - \cos^{-1}b = \cos^{-1}\{ab + \sqrt{(1-a^2)}\sqrt{(1-b^2)}\}.$$

167. To resolve $\tan^{-1}a$ into the sum of two angles of which α is one.

Let θ be the other angle; then $\tan^{-1}a = \alpha + \theta$;
$$\therefore\ a = \tan(\alpha+\theta) = \frac{\tan\alpha + \tan\theta}{1-\tan\alpha\tan\theta};$$
whence $$\tan\theta = \frac{a-\tan\alpha}{1+a\tan\alpha},\ \text{or}\ \theta = \tan^{-1}\left(\frac{a-\tan\alpha}{1+a\tan\alpha}\right).$$
$$\therefore\ \tan^{-1}a = \alpha + \tan^{-1}\left(\frac{a-\tan\alpha}{1+a\tan\alpha}\right).$$

168. By putting $b = a$ in §§ 165, 166, we obtain the following results which are often required:—
$$(1)\ 2\tan^{-1}a = \tan^{-1}\frac{2a}{1-a^2}.$$
$$(2)\ 2\sin^{-1}a = \sin^{-1}(2a\sqrt{1-a^2}).$$
$$(3)\ 2\cos^{-1}a = \cos^{-1}(2a^2-1).$$

169. By repeating the processes we can find the angles represented by $3\tan^{-1}a$, $3\sin^{-1}a$, $3\cos^{-1}a$, &c.

For example,
$$3\tan^{-1}a = \tan^{-1}a + 2\tan^{-1}a,$$
$$= \tan^{-1}a + \tan^{-1}\frac{2a}{1-a^2},\ \text{by (1)}$$
$$= \tan^{-1}\frac{a+\dfrac{2a}{1-a^2}}{1-\dfrac{a\cdot 2a}{1-a^2}},\ \text{by § 165,}$$
$$= \tan^{-1}\left(\frac{3a-a^3}{1-3a^2}\right).$$

Example 1.—Show that

$$\tan^{-1}\frac{3}{11}+\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{5}=\frac{\pi}{4}.$$

By § 165, $$\tan^{-1}\frac{3}{11}+\tan^{-1}\frac{1}{3}=\tan^{-1}\frac{\frac{3}{11}+\frac{1}{3}}{1-\frac{1}{11}}$$

$$=\tan^{-1}\frac{2}{3}.$$

Similarly $$\tan^{-1}\frac{2}{3}+\tan^{-1}\frac{1}{5}=\tan^{-1}\frac{\frac{2}{3}+\frac{1}{5}}{1-\frac{2}{15}}$$

$$=\tan^{-1}1=\frac{\pi}{4}.$$

Example 2.—Solve the equation

$$\sin^{-1}x+\cos^{-1}2x=\frac{\pi}{6};$$

$$\therefore\ \cos^{-1}2x=\frac{\pi}{6}-\sin^{-1}x \quad . \quad . \quad (1)$$

Let $\sin^{-1}x=y$; $\therefore\ \sin y=x$ and $\cos y=\sqrt{1-x^2}$,

then from (1) $2x=\cos\left(\frac{\pi}{6}-y\right)$

$$=\cos\frac{\pi}{6}\cdot\cos y+\sin\frac{\pi}{6}\sin y.$$

$$=\frac{\sqrt{3}}{2}\cdot\sqrt{1-x^2}+\frac{1}{2}\cdot x.$$

whence $3x=\sqrt{3(1-x^2)}$.

squaring, $3x^2=1-x^2$,

$$\therefore\ x=\pm\frac{1}{2}.$$

Example 3.—If $\cos^{-1}x+\cos^{-1}y+\cos^{-1}z=\pi$, show that

$$x^2+y^2+z^2+2xyz=1.$$

From the given relation

$$\cos^{-1} x + \cos^{-1} y = \pi - \cos^{-1} z.$$

$$\therefore \quad \cos\{\cos^{-1} x + \cos^{-1} y\} = \cos(\pi - \cos^{-1} z);$$

or
$$x y - \sqrt{(1 - x^2)(1 - y^2)} = -z.$$

$$\therefore \quad (x y + z)^2 = (1 - x^2)(1 - y^2);$$

or
$$x^2 + y^2 + z^2 + 2 x y z = 1.$$

EXAMPLES XXVII.

Prove the following identities:—

(1) $\tan^{-1}\dfrac{3}{4} + \tan^{-1}\dfrac{1}{7} = \dfrac{\pi}{4}$.

(2) $\sin^{-1}\dfrac{3}{5} + \sin^{-1}\dfrac{5}{13} = \tan^{-1}\dfrac{56}{33}$.

(3) $\tan^{-1}\dfrac{4}{3} = \dfrac{1}{2}\tan^{-1}\left(-\dfrac{24}{7}\right) = \dfrac{1}{3}\cos^{-1}\left(-\dfrac{117}{125}\right)$

(4) $\tan^{-1}\dfrac{1}{3} + \sin^{-1}\dfrac{1}{\sqrt{5}} = 45°$.

(5) $\tan^{-1}\dfrac{5}{6} + \tan^{-1}\dfrac{1}{11} = 45°$.

(6) $2\sin^{-1}\dfrac{3}{5} = \sin^{-1}\dfrac{24}{25} = \tan^{-1}\dfrac{24}{7}$.

(7) $2\cos^{-1}\dfrac{4}{5} = \cos^{-1}\dfrac{7}{25}$.

(8) $\dfrac{1}{2}\tan^{-1}\dfrac{24}{7} = \dfrac{1}{3}\cos^{-1}\left(-\dfrac{44}{125}\right)$.

(9) $\tan^{-1}\dfrac{1}{2} + \tan^{-1}\dfrac{1}{3} = 2\tan^{-1}\dfrac{1}{3} + \tan^{-1}\dfrac{1}{7}$.

(10) $\cot^{-1}\dfrac{1}{3} + \tan^{-1}\dfrac{1}{2} + \sin^{-1}\dfrac{7\sqrt{2}}{10} = 0$.

(11) $\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{2}{9}+\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{8}=\frac{\pi}{4}.$

(12) $\cos^{-1}\alpha-\cos^{-1}\beta=\sin^{-1}(\beta\sqrt{1-\alpha^2}-\alpha\sqrt{1-\beta^2}).$

(13) $$\tan^{-1}\frac{m-1}{m}+\tan^{-1}\frac{1}{2m-1}=n\pi+\frac{\pi}{4}$$
$$=\tan^{-1}\frac{m}{m+1}+\tan^{-1}\frac{1}{2m+1}.$$

(14) $\cos^{-1}\frac{1}{2}-\sin^{-1}\frac{1}{5}=\sin^{-1}\left(\frac{6\sqrt{2}-1}{10}\right).$

(15) $3\sin^{-1}\alpha=\sin^{-1}\alpha(3-4\alpha^2).$

Solve these equations:—

(16) $\cot^{-1}x+\sin^{-1}\frac{1}{\sqrt{5}}=\frac{\pi}{4}.$

(17) $\tan^{-1}x+\tan^{-1}(1-x)=2\tan^{-1}\sqrt{x-x^2}.$

(18) $\sin^{-1}x+\sin^{-1}(1-x)=\cos^{-1}x.$

(19) $\cot^{-1}\sec x-\cot^{-1}\sec 2x=\tan^{-1}(2\cos x+1).$

(20) $\sec^{-1}\frac{x}{a}-\sec^{-1}\frac{x}{b}=\sec^{-1}b-\sec^{-1}a.$

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(21) If $\sin^{-1}m+\sin^{-1}n=\frac{\pi}{2}$, prove that $\sin^{-1}m=\cos^{-1}n$ and that $m\sqrt{1-n^2}+n\sqrt{1-m^2}=1.$

(22) In a triangle right angled at C prove that
$$\tan^{-1}\frac{a}{b+c}+\tan^{-1}\frac{b}{a+c}=\frac{\pi}{4}.$$

(23) Find the value of
$$\cot^{-1}\frac{ab+1}{a-b}+\cot^{-1}\frac{bc+1}{b-c}+\cot^{-1}\frac{ca+1}{c-a}.$$

(24) Show that
$$\cos^{-1}\frac{1-a^2\cos 2\alpha-2a\sin\alpha}{1+a^2-2a\sin\alpha}-\cos^{-1}\frac{\cos 2\alpha+2a\sin\alpha-a^2}{1+a^2-2a\sin\alpha}=2\alpha.$$

MISCELLANEOUS EXAMPLES.

(1) Find the number of degrees subtended at the centre of a circle by an arc = ·357 times the radius, taking $\pi = 3\cdot1416$.

(2) If $\tan A = 2 - \sqrt{3}$, find the value of

$$\frac{3 \sin A - 4 \sin^3 A}{4 \cos^3 A - 3 \cos A}.$$

(3) Solve the triangle in which $a = 30$, $b = 30\sqrt{3}$, $A = 30°$.

(4) Prove that

$$\frac{(1 + \sin A - \cos A)^2}{1 + \sin A} + \frac{(1 - \sin A + \cos A)^2}{1 - \sin A} = 4.$$

(5) If, in the triangle A B C, $\tan \frac{A}{2}$, $\tan \frac{B}{2}$, $\tan \frac{C}{2}$ are in geometrical progression, show that $b(a + c) = a^2 + c^2$.

(6) Solve the equations—

1. $4 \sin^2 A + 3 \operatorname{cosec}^2 A = 7.$
2. $\cos 6 A + \cos 4 A = 0.$

(7) If $\tan \theta = \frac{b}{a}$, then $a \cos 2\theta + b \sin 2\theta = a$.

(8) Find the angle B in the triangle whose sides are $a = 40$, $b = 50$, $c = 60$.

Given $L \cos 27° 53' = 9\cdot9464040,$
diff. $1'' = \cdot0000669,$
$\log 2 = \cdot3010300.$

(9) Prove $\sin^{-1}\frac{1}{\sqrt{5}} + \cot^{-1} 3 = 45°$.

(10) Show that $\operatorname{cosec} A + \operatorname{cosec} 2A + \cot 2A = \cot \frac{A}{2}$.

(11) Prove that in a triangle A B C right-angled at C

$$\frac{\sin A + \sin B + 1}{\sin A} = \frac{a + b + c}{a}.$$

(12) At what time between 12 and half-past 12 o'clock is the angle between the hands of a watch equal to $2\frac{1}{2}$ radians?

(13) In an isosceles right-angled triangle a straight line is drawn from the middle point of one of the equal sides to the opposite angle. Show that it divides the angle into parts whose cotangents are 2 and 3.

(14) If $\sin \beta = m \sin (2\alpha + \beta)$,
then
$$\tan (\alpha + \beta) = \frac{1+m}{1-m} \tan \alpha.$$

(15) Solve the equations—

$$1.\ \cos 2A = \cos A.$$
$$2.\ \tan 2A = 3 \tan A.$$

(16) In any triangle if $\cos A = \cos B \cos C$, show that
$$\tan \frac{1}{2}(A+B) \tan \frac{1}{2}(A-B) = \tan^2 \frac{C}{2}.$$

(17) Show that
$$\frac{1 - \tan^2 A \tan^2 B}{\tan^2 A \tan^2 B} = \frac{\cos^2 A - \sin^2 B}{\sin^2 A \sin^2 B}.$$

(18) If $A + B + C = 360°$, then
$$\sin \frac{A}{2} + \sin \frac{B}{2} + \sin \frac{C}{2} = 4 \cos \frac{A}{4} \cos \frac{B}{4} \cos \frac{C}{4}.$$

(19) If two sides of a triangle be the roots of the equation $x^2 - 12x + 35 = 0$, and the cosine of the included angle be $-\frac{1}{10}$, show that the three sides of the triangle form an Arithmetical Progression.

(20) Prove that $\tan^{-1} \frac{4}{5} + 2 \tan^{-1} x = 45°$, if $\frac{2x}{1-x^2} = \frac{1}{9}$.

(21) In a triangle prove $\frac{\sin 3B - \sin 3C}{\cos 3C - \cos 3B} = \tan \frac{3A}{2}$.

(22) Find the length of an arc of 80° in a circle of 4 feet radius ($\pi = 3\frac{1}{7}$).

(23) In a triangle A B C prove that
$$(r_a - r)(r_b - r)(r_c - r) = 4 R r^2.$$

(24) A person standing at a point due south of a tower built on a horizontal plane, observes the altitude of the tower to be 60°. He then walks to a point B due west of A, and observes the altitude to be 45°, and again at C in A B produced he observes the altitude to be 30°. Show that B is midway between A and C.

(25) Solve $\cos^{-1}\sqrt{1-x^2}+\sin^{-1}(1-x)=\cos^{-1}x$.

(26) Two trees are 600 yards apart. The angles subtended at each tree by a church and the other tree are 45° and 30° respectively. Find their distances from the church.

(27) Prove that $2\sec 2A=\sec(45°+A)\sec(45°-A)$.

(28) Show that $\sin^2(45°-A)$, $\sin^2 45°$, and $\sin^2(45°+A)$ are in Arithmetical Progression, and that $\sec^2(45°-A)$, $\sec^2 45°$, $\sec^2(45°+A)$ are in Harmonical Progression.

(29) A man walks away from the foot of a leaning tower in a line vertically beneath it, and observes the top at an elevation of $57\frac{1}{2}°$. He then walks past the tower to an equal distance on the other side, and observes the elevation of the top to be $32\frac{1}{2}°$. Show that the tower makes an angle of 25° with the vertical.

(30) If $A+B+C=180°$, show that

$$\tan\frac{A+B}{2}+\tan\frac{B+C}{2}+\tan\frac{C+A}{2}$$
$$=\tan\frac{A+B}{2}\cdot\tan\frac{B+C}{2}\tan\frac{C+A}{2}.$$

(31) Solve the equations—

1. $\tan\frac{x}{2}=\operatorname{cosec}x$.

2. $\tan x(\tan 2x+\cot x)=2$.

(32) If r be the radius of the inscribed circle of a triangle, show that the product of the lengths of the three perpendiculars from the angles on the opposite sides will be

$$\frac{(a+b+c)^3}{a\,b\,c}\cdot r^3.$$

(33) Show that

$$\cot^{-1}2+\cot^{-1}4=\cot^{-1}3+\cot^{-1}5+\cot^{-1}7+\cot^{-1}21.$$

(34) Prove that in any triangle A B C

$$(1)\ \tan A=\frac{a\sin B}{\sin C-a\cos B}.$$

$$(2)\ \frac{\sin\left(\frac{A}{2}+B\right)}{\sin\frac{A}{2}}=\frac{b+c}{a}.$$

(35) A pole is fixed on the top of a mound, and the angles of elevation of the top and bottom of the pole are 60° and 30°. Show that the length of the pole is twice the height of the mound.

(36) In a triangle $A = 22\frac{1}{2}°$, $B = 45°$, $c = 220$ yds., show that its area is $2\frac{1}{2}$ acres.

(37) If $\tan^2\beta \cdot \cos^2\frac{\alpha+\gamma}{2} = \sin\alpha\sin\gamma$, show that $\tan\frac{\alpha}{2}$, $\tan\frac{\beta}{2}$, $\tan\frac{\gamma}{2}$, form a Geometrical Progression.

(38) Solve the equation

$$\tan^{-1}\frac{x-3}{x-4} - \tan^{-1}\frac{x-4}{x-3} = \tan^{-1}\frac{1}{x-7}.$$

(39) In a triangle right-angled at C, show that

$$\cot\frac{A}{2} + \cot\frac{B}{2} = \frac{2c}{a+b-c}.$$

(40) The area of any triangle is

$$\frac{2abc}{a+b+c} \cdot \cos\frac{A}{2} \cdot \cos\frac{B}{2} \cdot \cos\frac{C}{2}.$$

(41) The angles of a triangle are 45°, 60°, and 75°; the shortest side is $2\sqrt{2}$; prove that the other sides are $2\sqrt{3}$ and $\sqrt{2}(\sqrt{3}+1)$.

(42) Two sides of a triangle are 81·1 and 105·75, and the included angle is 47° 52′. Find the remaining angles, having given

$$\log 24{\cdot}65 = 1{\cdot}391817, \qquad L\cot 23°\,56' = 10{\cdot}352778,$$
$$\log 186{\cdot}85 = 2{\cdot}271493, \qquad L\tan 16°\,33' = 9{\cdot}473102.$$

(43) If I be the centre of the circle inscribed in the triangle A B C, prove that

$$\frac{AI^2}{bc} + \frac{BI^2}{ca} + \frac{CI^2}{ab} = 1.$$

(44) Given $\log_{10} 2 = {\cdot}30103$, find the value of

$$\log_{1000}\left(\frac{\tan 15° + 2\sin 60°}{\cos 45°}\right)^{\frac{10}{8}}.$$

(45) In any triangle show that

$$a\cos x = b\cos(x - C) + c\cos(x + B),\ x \text{ being any angle.}$$

(46) If in any triangle $b^2 - a^2 = (b^2 + a^2) \cos 2A$ the triangle is right-angled.

(47) If $\sin(\pi-\theta)\cos\left(\phi-\frac{3}{2}\pi\right) = \sec\left(\frac{3}{2}\pi-\theta\right)\operatorname{cosec}(\phi-2\pi)$, find $\sin\theta$ in terms of ϕ.

(48) If $\tan\theta = \dfrac{\sin\alpha\cos\beta}{\sin\beta+\cos\alpha}$, prove that

$$\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right) = \frac{\cos\left(\frac{\pi}{4}-\frac{\alpha+\beta}{2}\right)}{\sin\left(\frac{\pi}{4}-\frac{\alpha-\beta}{2}\right)}.$$

(49) Employ the formula

$$2\cos\frac{A}{2} = \pm\sqrt{1+\sin A} \pm \sqrt{1-\sin A}$$

to find $\cos 165°$.

(50) Show that $\sin 9°$ lies between $\cdot 156$ and $\cdot 157$.

(51) Two sides of a triangle are $5\cdot 5$ feet and $4\cdot 5$ feet, and include an angle of 120°. Find the angle opposite the greater given side; if $\log 3 = \cdot 4771213$, $L \tan 3° 18' = 8\cdot 7608719$, diff. for $10'' = \cdot 0002193$.

(52) If $A + B + C = 180°$, prove that $\sin(B + C)\cos A + \sin(C + A)\cos B + \sin(A + B)\cos C = 2\sin A\sin B\sin C$.

(53) Solve the equation—

$$\cos x + \cos 7x = \cos 4x.$$

(54) If $\tan A + \tan B + \tan C = \tan A\tan B\tan C$, prove that $A + B + C$ is a multiple of two right angles.

(55) Prove that in any triangle

$$a\sin(B - C) + b\sin(C - A) + c\sin(A - B) = 0.$$

(56) Prove that $\sin^{-1}\dfrac{1}{\sqrt{5}} + \cos^{-1}\dfrac{5}{\sqrt{26}} + \cot^{-1} 8 = \dfrac{\pi}{4}$.

(57) Show that $\cot\theta = \dfrac{\cos\theta + \sin 2\theta}{1 + \sin\theta - \cos 2\theta}$.

(58) Two sides of a triangle containing an acute angle are 5 in. and 7 in., and the area is 14 sq. in. Find the third side.

(59) A tower subtends an angle of 30° at a point 210 ft. from its base, and a steeple above subtends an angle of 15° at the same point. Find the heights of the tower and steeple.

(60) If $\cot A = 2 \tan B$, show that

$$\cos (A - B) = 3 \cos (A + B).$$

(61) Prove that $\sin (\alpha + 2\theta) \sin \alpha - \sin (\theta + 2\alpha) \sin \theta + \sin^2 \theta$ is independent of θ.

(62) From a circular board of 2 ft. radius and 3 in. thick, a circle of 21 in. radius is cut out. How many cubic inches are left? ($\pi = 3\frac{1}{7}$).

(63) Prove that in any triangle

$$\frac{a \cos A + b \cos B + c \cos C}{a + b + c} = \cos A + \cos B + \cos C - 1.$$

(64) Prove that

$$\tan \theta \tan 3\theta \tan 4\theta + \tan 3\theta + \tan \theta - \tan 4\theta = 0.$$

(65) In a triangle find a, given $b = 72$, $c = 56$, $A = 70°$.

$\log 2 = \cdot 30103$, $\log 3 = \cdot 47712$, $\log 7 = \cdot 84510$,
$L \cos 35° = 9 \cdot 91336$, $L \cos 35° \, 38' \, 10'' = 9 \cdot 90994$,
$\log 75 = 1 \cdot 87506$, $L \cos 54° \, 21' \, 50'' = 9 \cdot 76785$.

(66) In a triangle where C is a right angle, prove that—

1. $a b c = a^3 \cos A + b^3 \cos B$.
2. $b \sin^2 \frac{A}{2} + a \sin^2 \frac{B}{2} + 2 c \sin^2 \frac{A}{2} \sin^2 \frac{B}{2} = b \sin \frac{A}{2}$.

(67) Find the greatest value which $\sin \theta + \cos \theta$, and the least value which $\tan \theta + \cot \theta$, can have when θ lies between 0 and 90°.

(68) Prove that the product of $\sin A + \cos A$ and $\sin 2A + \cos 2A$ is $\sin 3A + \cos A$.

(69) In a quadrant of a circle another circle is inscribed. Prove that its area is $\frac{1}{3 + 2\sqrt{2}}$ of the area of the first circle.

(70) In the ambiguous case of the solution of triangles, if the angle A and the sides b and a are given, show that the two values of the third side are given by the equation

$$x^2 - 2b \cos A \, x + b^2 - a^2 = 0.$$

Hence find the condition that the triangle should be right-angled.

(71) Show that $\frac{\sin A + \sin 5A}{\sin 3A} = \frac{\sin 3A + \sin 7A}{\sin 5A}$

(72) Solve the equations—

$$\left.\begin{aligned} 4\sin^2\theta + 16\cos^2\phi &= 3 \\ \cos^2\theta - \cos\phi &= \frac{1}{2} \end{aligned}\right\}$$

(73) If in a triangle A = 3 B, prove that

$$\sin B = \frac{1}{2}\sqrt{\frac{3b-a}{b}}.$$

(74) In any triangle $\dfrac{r}{\sqrt{(r_b - r)(r_c - r)}} = \sin\dfrac{A}{2}$.

(75) Show that in any circle the chord of an arc of 108° is equal to the sum of the chords of arcs of 36° and 60°.

(76) If $\tan\theta = \dfrac{b\sin\phi}{a + b\cos\phi}$, show that $\tan(\phi - \theta)\dfrac{a\sin\phi}{b + a\cos\phi}$.

(77) In the ambiguous case of the solution of triangles, if C_1, C_2 be the two values of C when a, b, A are given, show that

$$\sin\frac{C_1 + C_2}{2} \cdot \sin\frac{C_1 - C_2}{2} = \cos A \cos B.$$

(78) D E F is the pedal triangle of the triangle A B C (*i.e.*, formed by joining the feet of perpendiculars from A, B, and C on the opposite sides). Prove that its perimeter is $a\cos A + b\cos B + c\cos C$, and find the radius of the circle circumscribing it.

(79) In any triangle prove that—

1. $r_b r_c + r_c r_a + r_a r_b = s^2$.
2. $4\left(\frac{s}{a} - 1\right)\left(\frac{s}{b} - 1\right)\left(\frac{s}{c} - 1\right)\left(\frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c}\right) = \frac{1}{R}$.

(80) Find the values of cos 12°, sin 42°, cot 660°, sec 840°.

(81) Deduce the formulæ

$$2\sin\frac{A}{2} = \pm\sqrt{(1 + \sin A)} \pm \sqrt{1 - \sin A}$$

$$2\cos\frac{A}{2} = \pm\sqrt{(1 + \sin A)} \mp \sqrt{1 - \sin A}$$

from the formula $\cos\dfrac{A}{2} = \sqrt{\dfrac{1 + \cos A}{2}}$.

(82) Resolve $\sin^2 x\,(4 + \sin^2 x) + 4\cos^2 x$ into two factors.

(83) Given $\sin^3\theta\cos^5\theta = A\sin 8\theta + B\sin 6\theta + C\sin 4\theta + D\sin 2\theta$; find the values of A, B, C, and D.

(84) Prove that

$$\tan^{-1}\frac{\tan a}{\tan a+1}+\tan^{-1}\frac{1}{2\tan a+1}=n\pi+\frac{\pi}{4}.$$

(85) Prove geometrically that $\tan(45^\circ+A)=\dfrac{1+\tan A}{1-\tan A}$.

(86) Eliminate θ and ϕ from the equations $a\sin\theta+b\sin\phi=c$, $a\cos\theta=b\cos\phi$, $ab(\tan\theta+\tan\phi)=c^2$.

(87) Solve the equation—

$$\left(\frac{\sec^2 x}{\sec a}-\frac{\tan^2 x}{\tan a}\right)(\sec a+\tan a)=1.$$

(88) Show that

$$\frac{3\cos 3\theta-2\cos\theta-\cos 5\theta}{\sin 5\theta-3\sin 3\theta+4\sin\theta}=\tan 2\theta.$$

(89) Prove that $\cot 20^\circ+\cot 80^\circ+\cot 140^\circ=\sqrt{3}$

(90) Eliminate θ from the equations

$x\cos\theta+y\sin\theta=a\sin 2\theta$, $x\sin\theta-y\cos\theta=a\cos 2\theta$.

(91) In any triangle whose area is Δ, show that

$$r^2(ab+bc+ca)=\Delta^2+r^3(r+4R).$$

(92) Show that

$$\sin 84^\circ\cos 54^\circ-\cos 54^\circ\sin 24^\circ+\sin 24^\circ\sin 84^\circ=\frac{3}{4}.$$

(93) If a quadrilateral be inscribed in a circle, and circumscribe another circle, its area $=\sqrt{abcd}$; a, b, c, d being its sides.

(94) Prove that

$$4\tan^{-1}\frac{1}{5}-\tan^{-1}\frac{1}{70}+\tan^{-1}\frac{1}{99}=\frac{\pi}{4}.$$

(95) In any triangle A B C, prove that

$$\frac{c^2-b^2}{a^2}\sin A+\frac{a^2-c^2}{b^2}\sin B+\frac{b^2-a^2}{c^2}\sin C$$
$$=4\sin\frac{B-C}{2}\cdot\sin\frac{C-A}{2}\cdot\sin\frac{A-B}{2}.$$

(96) Prove that if $\alpha+\beta+\gamma=45^\circ$

$$\frac{1+\tan\alpha}{1-\tan\alpha}+\frac{1+\tan\beta}{1-\tan\beta}+\frac{1+\tan\gamma}{1-\tan\gamma}$$
$$=\frac{(1+\tan\alpha)(1+\tan\beta)(1+\tan\gamma)}{(1-\tan\alpha)(1-\tan\beta)(1-\tan\gamma)}.$$

(97) Eliminate θ from the equations

$$x \cos\theta + y \sin\theta = a\,(\cos^2\theta + 3\sin^2\theta),$$
$$x \sin\theta + y \cos\theta = a\,(\sin^2\theta + 3\cos^2\theta).$$

(98) Prove that the distance between the centres of the circumscribed and an escribed circle of a triangle

$$= (\mathrm{R}^2 + 2\,\mathrm{R}\,r_1)^{\frac{1}{2}}.$$

(99) Prove that in any plane triangle

$$\sin^2\frac{\mathrm{A}}{2} + \sin^2\frac{\mathrm{B}}{2} + \sin^2\frac{\mathrm{C}}{2} + 2\sin\frac{\mathrm{A}}{2}\sin\frac{\mathrm{B}}{2}\sin\frac{\mathrm{C}}{2} = 1.$$

(100) Show that the magnitude of the area of any quadrilateral in terms of the four sides a, b, c, d, and the angle θ, between the diagonals $= \dfrac{1}{4}(a^2 - b^2 + c^2 - d^2)\tan\theta$.

(101) The cotangents of the angles of a triangle are α, β, γ, and the tangents of the angles which the sides make with any straight line are l, m, n, taken symmetrically; show that

$$mn\,\beta\gamma + nl\,\gamma\alpha + lm\,\alpha\beta + 1 = 0.$$

(102) Eliminate θ, ϕ from the equations

$$\cos\theta\,\mathrm{cosec}^2\phi = \cos\phi\,\mathrm{cosec}^2\theta = ab^{-1},$$
$$a^2\cos\theta\cos\phi = c.$$

(103) From three points A, B, C in the same straight line such that BC $= a$, AC $= b$, AB $= c$, the altitudes of a balloon are simultaneously observed to be $\cot^{-1}\alpha$, $\cot^{-1}\beta$, $\cot^{-1}\gamma$; show that the height of the balloon above the straight line is

$$\{abc\,(a\,\alpha^2 - b\,\beta^2 + c\,\gamma^2)^{-1}\}^{\frac{1}{2}}.$$

EXERCISES IN TRIGONOMETRY.

Take $\pi = 3\frac{1}{7}$ unless the contrary is stated.

Exercise 1.

(1) A railway train is travelling on a curve of half a mile radius at the rate of 20 miles per hour; through what angle does it turn in 11 seconds?

(2) Prove that $\sec B - \tan B = \dfrac{\cos B}{1 + \sin B}$.

(3) Prove that $\dfrac{\tan 20^\circ \sec 80^\circ + \cot 10^\circ \operatorname{cosec} 70^\circ}{\tan 80^\circ \sec 20^\circ + \cot 70^\circ \operatorname{cosec} 10^\circ} = 1$.

(4) The portions of a flagstaff above and below a mark on it subtend angles of 30° at a point on the horizontal plane on which it stands. If the mark be 20 ft. above the ground find the height of the flagstaff.

(5) Solve the equation $\tan^2 \theta = 3 \operatorname{cosec}^2 \theta - 1$.

(6) If $\tan \theta = \dfrac{a}{b}$, find the value of $\dfrac{a \sin \theta + b \cos \theta}{a \sin \theta - b \cos \theta}$.

(7) If $\tan A + \sin A = m$, and $\tan A - \sin A = n$, find $\cos A$ and $\sin A$ and show that $(m^2 - n^2)^2 = 16\, m\, n$.

(8) In a triangle with C a right angle, prove that

$$\frac{\tan A + \sec A}{\cot A + \operatorname{cosec} A} = \frac{c^2 + a\,c - b^2}{c^2 + b\,c - a^2}.$$

(9) If $\cos x = \dfrac{\cos A}{\sin C}$ and $\cos (90^\circ - x) = \dfrac{\cos B}{\sin C}$, prove that $\sin^2 A + \sin^2 B + \sin^2 C = 2$.

(10) Having given $\dfrac{\sin A}{\sin B} = \dfrac{\sqrt{2}}{\sqrt{3}}$ and $\dfrac{\tan A}{\tan B} = \dfrac{1}{\sqrt{3}}$; find A and B.

Exercise 2.

(1) The sides of two regular polygons subtend angles at their centres whose circular measures are the roots of the equation $105 x^2 - 29 \pi x + 2 \pi^2 = 0$. Find the number of sides in each polygon.

(2) In a triangle right-angled at C, show that

$$\sin A + \sin B + \sin C$$

is known when the hypothenuse and the sum of the sides are given.

(3) Prove that $\dfrac{1 + \cot 60^\circ}{1 - \cot 60^\circ} = \left\{\dfrac{1 + \cos 30^\circ}{1 - \cos 30^\circ}\right\}^{\frac{1}{2}}$.

(4) Three angles of a quadrilateral are 60°, 60°, and $\dfrac{5\pi}{6}$ respectively, find the number of *degrees* in the fourth.

(5) If $\sin^2 A \operatorname{cosec}^2 B + \cos^2 A \cos^2 C = 1$, show that

$$\pm \sin C = \tan A \cot B.$$

(6) Eliminate θ between the equations $\sec\theta = m + \tan\theta$, $n \sec\theta = 1 - n \tan\theta$.

(7) Solve the equation $\sin\theta \tan\theta = \dfrac{1}{\sqrt{2}}$.

(8) Given $\sin\theta = \dfrac{2ab}{a^2 + b^2}$ find $\cos\theta$ and $\tan\theta$.

(9) Find the area of a right-angled triangle A B C, when $\tan A = \dfrac{1}{3}$ and the hypothenuse $C = \sqrt{1000}$ in.

(10) C D is the perpendicular from the right angle C of a triangle on the hypothenuse A B; A C = 108 yds., C B = 144 yds.; find C D, B D, D A.

Exercise 3.

(1) One exterior angle of a triangle is half as large again as one of the interior and opposite angles and its supplement is half as large again as the other. Find the angles of the triangle.

(2) Express in degrees the angle whose circular measure is ·7854 ($\pi = 3{\cdot}1416$).

(3) Solve the equations (1) $\sin 2\theta = \sin\theta$.
(2) $\sin\theta + \sin 2\theta = \cos\theta + \cos 2\theta$.

(4) Prove that (1) $2 \operatorname{cosec} 4A + 2 \cot 4A = \cot A - \tan A$.

$$(2)\ (1 - \cot A)^2 = \frac{2(1 - \sin 2A)}{1 - \cos 2A}.$$

(5) Show that $(\sec 30^\circ + \tan 30^\circ)(\operatorname{cosec} 60^\circ + \tan 60^\circ) = 5$.

(6) If $x \cos B + y \cos A = t$ and $x \sin B - y \sin A = 0$, prove that $\dfrac{x}{\sin A} = \dfrac{y}{\sin B} = \dfrac{t}{\sin (A + B)}$.

(7) Show that $\tan 70^\circ + \tan 20^\circ = 2 \sec 50^\circ$.

(8) Find the values of $\tan(-480^\circ)$, $\cot 660^\circ$, $\sec 840^\circ$, $\operatorname{cosec} 900^\circ$.

(9) Show that $\sec 22\frac{1}{2}^\circ = 1 \cdot 0824$ nearly.

(10) A man is flying a kite whose string is stretched and makes an angle of $67\frac{1}{2}^\circ$ with the horizontal plane. If 150 yds. of string are let out, find the height of the kite above the horizontal plane.

Exercise 4.

(1) Find the magnitude of an angle of a regular polygon of 36 sides in degrees and in circular measure.

(2) Solve the equations (1) $\sec \theta \operatorname{cosec} \theta - \cot \theta = 2$.
(2) $\sin 2\theta - \cos 2\theta = 1$.

(3) Find the perimeter of a right-angled triangle A B C, given that the perpendicular from the right angle at C on A B is 60 ft. and that $2\,BC = AB$.

(4) Show that if $\sin (A - C) \cos B + \sin (B - C) \cos A = 0$, then $\tan A$, $\tan C$, and $\tan B$ are in arithmetical progression.

(5) In any triangle show that

$$(1)\ \frac{\sin (A - B)}{\sin (B - C)} = \frac{(a^2 - b^2) \sin A}{(b^2 - c^2) \sin C}.$$

$$(2)\ \frac{\sin A + \sin C}{\cos A + \cos C} = \frac{\cos C - \cos A}{\sin A - \sin C} = \cot \frac{B}{2}.$$

(6) Show that $\sin 30^\circ\, 1' = \cos 1' - \sin 29^\circ\, 59'$.

(7) Prove that $\tan 7^\circ\, 30' = (\sqrt{3} - \sqrt{2})(\sqrt{2} - 1)$.

(8) Show that $\tan^{-1}(2 + \sqrt{3}) - \tan^{-1}(2 - \sqrt{3}) = 60^\circ$.

(9) Prove that an inch will subtend an angle of $1''$ very nearly at a distance of 3 miles.

(10) Two objects are seen in the same direction to the S.E., and when the observer has moved 8 miles due S. the one bears N.E. and the other due E.; how far are they apart?

EXERCISE 5.

(1) One angle of a triangle is $\frac{\pi}{4}$, another $\frac{1}{4}$ of a radian, express the third angle in degrees, &c.

(2) Solve the equations (1) $\sin^2 \theta + \cos^2 (90 - \theta) = 1$.
(2) $1 + 2 \sin 4 \theta = 4 \sin 3 \theta \cos \theta$.

(3) If $\cos \phi = \frac{5}{13}$ find $\sin \frac{\phi}{2}$, $\sin \phi$, $\sin 2 \phi$.

(4) A person on the bank of a river observes the angular elevation of the top of a tree on the opposite bank to be 60° When he retires 100 ft. from the edge of the river the angle is 30°. Find the height of the tree and the breadth of the river.

(5) In a right-angled triangle with a right angle at C, if $a = 576 \cdot 12$, $c = 873 \cdot 14$, find A and B:

$\log 5 \cdot 7612 = \cdot 7605054$, $\quad$ $L \sin 41° 17' = 9 \cdot 8194012$,
$\log 8 \cdot 7314 = \cdot 9410839$, $\quad$ $L \sin 41° 18' = 9 \cdot 8195450$.

(6) In any triangle prove that

$$a \cos 2 B + 2 b \cos A \cos B + a \cos 2 C + 2 c \cos A \cos C = 0.$$

(7) Prove that $\tan^{-1} \frac{2}{11} + 2 \tan^{-1} \frac{1}{7} = \tan^{-1} \frac{1}{2} = \frac{\pi}{4} - \tan^{-1} \frac{1}{3}$.

(8) Show that $\dfrac{\sin 2 a + \cos 2 a}{\cos a - \sin a - (\cos 3 a - \sin 3 a)} = \dfrac{1}{2} \operatorname{cosec} a$.

(9) Find the value of $\cos 3000°$, $\tan 2565°$, $\sec \left(\frac{-11 \pi}{4}\right)$, $\operatorname{cosec} \frac{23 \pi}{6}$.

(10) Given $a \tan^3 \theta = b$, find the values of $a \sin \theta + b \cos \theta$, and of $a \sec \theta + b \operatorname{cosec} \theta$.

EXERCISE 6.

(1) If an equilateral triangle, a square, and a regular hexagon be described about the same circle, their sides will be in Geometrical Progression.

(2) Calculate approximately the distance at which a globe $5\frac{1}{2}$ in. in diameter would subtend an angle of $6'$.

(3) Prove that—

$$1.\ 1 + \tan 2A \tan A = \sec 2A.$$
$$2.\ \sin 105^\circ + \cos 105^\circ = \cos 45^\circ.$$

(4) A man stands on the top of a wall 12 ft. high, and observes the angle of elevation 30° of the top of a telegraph post; he then descends from the wall, and finds that the angle of elevation is now 45°; prove that the height of the post exceeds the height of the man's eye above the ground by $6(\sqrt{3} + 3)$ ft.

(5) In a triangle $b = 5$, $c = 4$, $A = 60^\circ$; find B and C, having given $\log 3 = \cdot 477121$, $L \tan 10^\circ 53' = 9\cdot 283907$, $L \tan 10^\circ 54' = 9\cdot 284525$.

(6) Solve the equations—

$$1.\ \sin\left(x + \frac{\pi}{4}\right) \sin\left(x - \frac{\pi}{4}\right) = \sin^2 \frac{\pi}{6}.$$
$$2.\ \sec x = 2 \sin^2 x + \cos x.$$

(7) In any triangle prove—

$$1.\ \cot A = \frac{b}{a} \operatorname{cosec} C - \cot C.$$

$$2.\ \frac{\cos(A - B)\cos C - \cos(A - C)\cos B}{1 + \cos(A - C)\cos B} = \frac{b^2 - c^2}{a^2 + c^2}.$$

(8) Solve the equation—

$$8^x . 125^{2-x} = 2^{4x+3} . 5^x,$$

given $\log 2 = \cdot 30103.$

(9) Show that the length of the line C D, which is drawn to the side A B of any triangle so as to bisect the angle C, is

$$\frac{2ab}{a + b} \cos \frac{C}{2}.$$

(10) In any triangle prove that

$$r_a + r_b + r_c = 2R\left(\cos^2 \frac{A}{2} + \cos^2 \frac{B}{2} + \cos^2 \frac{C}{2}\right).$$

Exercise 7.

(1) An arc of 120° on one circle is equal to the whole circumference of another. Compare the area of a square inscribed in the larger circle with that of a square circumscribing the smaller one.

(2) Prove that—

1. $\tan(A + 30°)\tan(A - 30°) = \dfrac{1 - 2\cos 2A}{1 + 2\cos 2A}$.

2. $\tan^{-1}\dfrac{\cos\phi}{1 - \sin\phi} - \tan^{-1}\dfrac{1 - \sin\phi}{\cos\phi} = \phi$.

(3) In a triangle, given $A = 62° 39'$, $B = 75° 32'$, $c = 382 \cdot 05$, find a, given

$L \sin 62° 39' = 9 \cdot 9485189$, $L \sin 41° 49' = 9 \cdot 8239625$,
$\log 38205 = 4 \cdot 5821088$, $\log 50894 = 4 \cdot 7066666$,
$\log 50893 = 4 \cdot 7066581$.

(4) Eliminate θ from $\cos(\phi - \theta + a)\cos(\theta - a) = b$, $\cos(\phi - \theta - a)\cos(\theta + a) = c$.

(5) Two objects, P, Q, are observed from a ship to be at the same instant in a line 15° to the east of north. When the ship has sailed N.W. for five miles, P is due E. and Q is N.E. Find the distance P Q.

(6) Solve the equations—

1. $\sin\theta + \cos\theta = 4\cos^2\theta\sin\theta$.
2. $\sin(a - \theta) = \cos(a + \theta)$.
3. $\dfrac{1}{(\sqrt{1 - \sin\theta}) - 1} + \dfrac{1}{(\sqrt{1 + \sin\theta}) - 1} = \dfrac{1}{\sin\theta}$.

(7) The diagonals of a quadrilateral are a and b, and the angle they include is θ. Show that the area of the quadrilateral is $\frac{1}{2} a b \sin\theta$.

(8) In a right-angled triangle, C being the right angle, prove that $\cos(2A - B) = \dfrac{a}{c^3}(3c^2 - 4a^2)$, and that the area $= \dfrac{1}{4} c^2 \sin 2A$.

(9) The sides of a triangle are 9, 7, 4 ft. Find the sines of the angles.

(10) If $\alpha + \beta + \gamma = \pi$, prove that

$$\sin^2\gamma = \cos^2\alpha + \cos^2\beta + 2\cos\alpha\cos\beta\cos\gamma.$$

Exercise 8.

(1) An arc of a circle whose radius is 7 in. subtends an angle of 15° 39′ 57″ at the centre of a circle. What angle will an arc of the same length subtend at the centre of a circle of radius 2 in.?

(2) What are the proper signs for the ambiguities in the formula $2\cos\frac{A}{2} = \pm\sqrt{1+\sin A} \pm \sqrt{1-\sin A}$, (1) when A lies between 570° and 580°, (2) when A lies between − 570° and − 580°?

(3) A lake is bounded by a vertical cliff, whose height h is equal to the breadth of the lake. From a balloon above the lake the height of the cliff and the breadth of the lake both subtend the same angle a. Show that the height of the balloon is $\frac{h}{2}(\sin a + \cos a)\operatorname{cosec} a$.

(4) Solve the equations—

1. $4\sec^2\theta - 5\tan^2\theta = 1.$
2. $\cos x \cos 3x = \cos 2x \cos 6x.$
3. $(1 - \tan\theta)(1 + \sin 2\theta) = (1 + \tan\theta).$

(5) Show that if $\tan\theta + \cot\theta + 2 = 0$, then $\sin\theta + \cos\theta = 0$.

(6) In any triangle prove that—

1. $\dfrac{a\cos B - b\cos A}{c} = \dfrac{\sin^2 A - \sin^2 B}{\sin^2(A+B)}.$

2. $$(b-c)(s-a)\cos^2\frac{A}{2} + (c-a)(s-b)\cos^2\frac{B}{2} + (a-b)(s-c)\cos^2\frac{C}{2} = 0.$$

3. $(b^2 - c^2)\cot A + (c^2 - a^2)\cot B + (a^2 - b^2)\cot C = 0.$

(7) A circle of radius r is inscribed in a sector of a circle of radius a, the chord of the sector being $2c$. Show that

$$\frac{1}{r} = \frac{1}{a} + \frac{1}{c}.$$

(8) Trace the changes in sign and magnitude of (1) $\sin\theta\cos\theta$, (2) $\dfrac{\cos\theta - \sin\theta}{\cos\theta + \sin\theta}$, as θ changes from 180° to 270°.

(9) The sides of a triangle are 5, 12, and 13. Find all the angles, having given

$$\log 2 = \cdot 3010300, \quad L \tan 11^\circ\, 18'\, 30'' = 9 \cdot 3009670,$$
$$L \tan 11^\circ\, 18'\, 40'' = 9 \cdot 3010764.$$

(10) Prove that

$$\sin^{-1}\sqrt{\frac{x}{a+x}} = \tan^{-1}\sqrt{\frac{x}{a}} = \frac{1}{2}\cos^{-1}\frac{a-x}{a+x}.$$

Exercise 9.

(1) The moon's distance from the earth is 60 times the earth's radius. Show that the earth's radius subtends at the moon an angle of $\cdot 95^\circ$.

(2) Prove that—

1. $(\sin A + \sin B)^2 + (\cos A + \cos B)^2 = 4 \cos^2 \dfrac{A-B}{2}.$

2. $\dfrac{\cot(60^\circ + A)}{\tan(60^\circ - A)} = \dfrac{2 - \sec 2A}{2 + \sec 2A}.$

(3) If $\tan\dfrac{\theta}{2} = \dfrac{(1-\sin\beta)(1-\cos\alpha)}{\sin\alpha\cos\beta}$, show that

$$\tan\theta = \frac{\sin\alpha\cos\beta}{\sin\beta + \cos\alpha}.$$

(4) The sides of a triangle are 2, $\sqrt{6}$, and $1 + \sqrt{3}$. Find the angles.

(5) Find A and B from the equations—

$$\cos\frac{A+B}{2} \cdot \cos\frac{A-B}{2} = \frac{\sqrt{3}-1}{4}, \quad \cos A - \cos B = \frac{\sqrt{3}+1}{2}.$$

(6) Given $\sin\alpha = m\sin\beta$, $\tan\alpha = n\tan\beta$, prove that

$$\cos^2\alpha = \frac{m^2-1}{n^2-1}.$$

(7) Solve the equations—

1. $3\tan 2x \cot x = 2\sqrt{3} + 3.$
2. $\cos x + \sin x = 2\sqrt{2}\sin x \cos x.$

(8) In a triangle, having given $a = 456 \cdot 12$, $b = 296 \cdot 86$, $C = 74^\circ\, 20'$, find A and B.

$$\log 1 \cdot 5926 = \cdot 2021067, \quad \log 7 \cdot 5298 = \cdot 8767834,$$
$$L \cot 37^\circ\, 10' = 10 \cdot 1202593, \quad L \cot 74^\circ\, 25' = 9 \cdot 4454352,$$
$$L \cot 74^\circ\, 24' = 9 \cdot 4459232.$$

(9) The area of a regular polygon inscribed in a circle is to that of the corresponding circumscribed polygon as 3 : 4. Find the number of sides.

(10) If α, β, γ denote the distances from the angular points of a triangle to the points of contact of the inscribed circle, show that its radius $= \left(\frac{\alpha\beta\gamma}{\alpha+\beta+\gamma}\right)^{\frac{1}{2}}$.

Exercise 10.

(1) The number of sides of one regular polygon exceeds that of another by 1, and an angle of the first exceeds one of the second by 4°. Find the number of sides in each.

(2) At what distance from the eye must a circular disc of 6 in. diameter be placed so as just to cover the moon, whose apparent angular diameter is 31′?

(3) Show that—

1. $\{\tan(A + 45°) + \tan(A - 45°)\} \div \{\cot(A + 45°) + \cot(A - 45°)\} = -1.$

(4) $1+\sin^2 A \cot^2 A - 2 \sin A \cot A + \cos^2 A \tan^2 A = 4 \sin^2 \frac{A}{2}.$

(5) If a b c are the sides of a triangle A B C, and $\cos x = \frac{a}{b+c}$, $\cos y = \frac{b}{c+a}$, $\cos t = \frac{c}{a+b}$, prove that—

1. $\tan^2 \frac{x}{2} + \tan^2 \frac{y}{2} + \tan^2 \frac{t}{2} = 1.$

2. $\tan \frac{x}{2} \cdot \tan \frac{y}{2} \cdot \tan \frac{t}{2} = \tan \frac{A}{2} \cdot \tan \frac{B}{2} \cdot \tan \frac{C}{2}.$

(6) Show that the angles of a triangle whose sides are $2\sqrt{3}$, $2\sqrt{2}$, and $\sqrt{6} + \sqrt{2}$ are in A. P.

(7) If a, ar, ar^2, be the sides of a triangle, show that r lies between $\frac{\sqrt{5}-1}{2}$ and $\frac{\sqrt{5}+1}{2}$.

(8) A man walking towards a church observes the nave to be α in elevation, and the spire behind it β. After walking a further, he sees the nave hide the spire at an elevation of 2α. Find the height of the spire.

(9) Given A = 30°, b = 12 in., find the values of a which make the triangle (1) right-angled (2) isosceles.

(10) Find the relations between the sides of a triangle when the angles are in arithmetical progression and the sum of the squares of their sines equal to 2.

Exercise 11.

(1) The apparent angular diameter of the sun is 30′. A planet is seen to cross its disc in a straight line at a distance from the centre equal to $\frac{3}{5}$ of the radius. Prove that the angle subtended at the earth by the part of the planet's path projected on the sun is $\frac{\pi}{450}$.

(2) Solve the equations—

1. $\operatorname{cosec} 2x = \cot x$.
2. $\cos 5\theta + \cos 3\theta = \sqrt{3} \cos \theta$.
3. $\tan^3 \theta - \sec^2 \theta = 4 \tan^2 \theta - 5 \tan \theta$.

(3) Show that

1. $\dfrac{\operatorname{cosec} 2A}{1 + \operatorname{cosec} 2A} = \dfrac{1 + \tan^2 A}{(1 + \tan A)^2}$.
2. $\cos 47° - \cos 61° - \cos 11° + \cos 25° - \sin 7° = 0$.
3. $\tan^{-1} \dfrac{4 + \sqrt{7}}{4 - \sqrt{7}} - \tan^{-1} \dfrac{4 - \sqrt{7}}{4 + \sqrt{7}} = \tan^{-1} \dfrac{8\sqrt{7}}{9}$.

(4) If in any triangle $\sin^2 A (\sin B - \sin C) + \sin^2 B (\sin C - \sin A) + \sin^2 C (\sin A - \sin B) = 0$, show that the triangle is isosceles.

(5) Prove that

1. $\dfrac{1 + \sin \theta + \cos \theta}{1 + \sin \theta - \cos \theta} = \cot \dfrac{\theta}{2}$.
2. $\{\cos \beta - \cos \gamma + \sin (\beta - \gamma)\}^2$
$= 2 \{1 - \cos (\beta - \gamma)\} (1 - \sin \beta) (1 - \sin \gamma)$.

(6) Two adjacent sides of a triangle are 55 and 40, and the angle opposite the greater side is 54° 10′, find the angle opposite the less.

$L \sin 54° 10' = 9{\cdot}9088727$, $L \sin 36° 7' = 9{\cdot}7704332$,
diff. for $1' = {\cdot}0001731$,
$\log 2 = {\cdot}30103$, $\log 22 = 1{\cdot}3424227$.

(7) Show that

$$1.\ \frac{1}{4}\tan^{-1}\frac{7}{24}+\tan^{-1}\frac{1}{3}=\frac{\pi}{8}.$$

$$2.\ \frac{1}{3}\tan^{-1}\frac{11}{2}+\frac{1}{4}\tan^{-1}\frac{24}{7}=\frac{\pi}{4}.$$

(8) In a triangle if $\tan A = 1$ and $\tan B = 2$, show that $\tan C = 3$.

(9) D E F are the feet of the perpendiculars from A B and C on the opposite sides. Prove that the area of the triangle D E F = 2 cos A cos B cos C times that of the original triangle.

(10) Show that

$$\cos^2\theta+\cos^2(\alpha+\theta)-2\cos\alpha\cos\theta\cdot\cos(\alpha+\theta)=\sin^2\alpha.$$

Exercise 12.

(1) The Arctic Expedition of 1875 reached latitude 82° 20′; how far was this from the North Pole, assuming the radius of the earth to be 4000 miles?

(2) Prove that

$$1.\ \tan\frac{A}{2}=\sqrt{\frac{2\operatorname{cosec}2A-\operatorname{cosec}A}{2\operatorname{cosec}2A+\operatorname{cosec}A}}.$$

$$2.\ 2\cos\theta=\sqrt{2+\sqrt{(2+2\cos4\theta)}}.$$

(3) Solve the equations—

$$1.\ \frac{1}{2}\tan\frac{x}{2}=\operatorname{cosec}x-\sin x.$$

$$2.\ \sin3\theta=2\sin\theta.$$

$$3.\ \cos nx+\cos(n-2)x=\cos x.$$

(4) Prove that the area of any triangle A B C

$$=\frac{(a\cos A+b\cos B)^2-c^2\cos^2 C}{8\cos A\cos B\sin C}.$$

(5) In any triangle, show that

$$\frac{r_a+r_b}{r_c}\tan^2\frac{C}{2}+\frac{r_b+r_c}{r_a}\tan^2\frac{A}{2}+\frac{r_c+r_a}{r_b}\tan^2\frac{B}{2}=2.$$

(6) Given

$$\sin(\beta+\theta)+\sin\alpha(\sin^2\beta-\sin^2\alpha)+\sin\{2\alpha-\beta+\theta\}$$
$$=(\cos^2\beta-\cos^2\alpha)\sin(\alpha+2\theta)$$

show that

$$\sec\theta=\sin(\alpha+\beta)\tan(\alpha-\beta).$$

(7) Prove that

$$\sin^{-1}\left(\frac{x-a+\beta}{2\beta}\right)^{\frac{1}{2}} = \frac{1}{2}\cos^{-1}\frac{a-x}{\beta}.$$

(8) In any triangle, if

$$\tan\frac{A}{2} : \tan\frac{B}{2} : \tan\frac{C}{2} :: 6 : 3 : 2,$$

show that

$$a : b : c :: 5 : 4 : 3.$$

(9) If θ be the acute angle between two adjacent sides of a rhombus, show that $\cos\theta = \dfrac{x^2-y^2}{x^2+y^2}$ where x and y are the lengths of the two diagonals.

(10) Show that

$$16\sin^3\theta\cos^2\theta = 2\sin\theta - \sin 5\theta + \sin 3\theta.$$

Exercise 13.

(1) The length of an arc of 45° in one circle being equal to that of 60° in another, find the circular measure of the angle that would be subtended at the centre of the first by an arc equal to the radius of the second.

(2) Prove that

1. $\cos(A+B)\cos(A-B) - \cos(B+C)\cos(B-C) + \cos(A+C)\cos(A-C) = \cos 2A.$
2. $\dfrac{\sin^2 2A - 4\sin^2 A}{\sin^2 2A + 4\sin^2 A - 4} = \tan^4 A.$

(3) Show that

$$3\sin^{-1}\frac{4}{5} = 2\sin^{-1}\frac{24}{25} - \frac{1}{2}\cos^{-1}\left(-\frac{7}{25}\right).$$

(4) Solve the equations

1. $\dfrac{\sin^4\theta}{\sin^2 a} + \dfrac{\cos^4\theta}{\cos^2 a} = 1.$
2. $\sin^2\left(\dfrac{\pi}{4} - \theta\right) + \sin^2 2\theta = \dfrac{3}{4}.$

(5) In a triangle prove that

1. $c^2 = (a+b)^2 \sin^2 \frac{C}{2} + (a-b)^2 \cos^2 \frac{C}{2}$.

2. $\cos A + \cos B = 2\,\frac{(a+b)}{c} \sin^2 \frac{C}{2}$.

3. The area $= \frac{1}{4}(a^2+b^2-c^2) \cot \frac{A+B-C}{2}$.

(6) An observer in a balloon when it is one mile high, observes the angle of depression of an object on the ground to be 30°. After ascending vertically for 25 minutes he finds the angle of depression of the same object to be 60°. Find the rate of ascent of the balloon in miles per hour.

(7) If $3 \sin^2 B + 2 \sin^2 A = 1,$

and $3 \sin 2B = 2 \sin 2A,$

prove that $\sin (A - B) = \frac{\sqrt{6}}{9}$.

(8) Transform $\sin x + \sin 3x + \sin 9x - \sin 5x$ into a product.

(9) Show that the equation $\sin \theta \sin (2\alpha + \theta) + n \cos^2 \alpha = 0$ cannot hold unless $\sec^2 \alpha$ is greater than $1 + n$.

(10) The distances between the centres of the escribed circles of a triangle being l, m, n, show that $\frac{l^2}{r_b+r_c} = \frac{m^2}{r_c+r_a} = \frac{n^2}{r_a+r_b} = 4R$.

Exercise 14.

(1) The circumferences of two circles are divided into a and b parts respectively. If α and β be the angles subtended at the centres of the two circles respectively by the divisions, show that if $\alpha + \beta = \frac{4\pi}{c}$ then a, c, b are in Harmonical Progression.

(2) If $\cos A = \tan B$, $\cos B = \tan C$, $\tan C = \tan A$, show that $\sin A = \sin B = \sin C = 2 \sin 18°$.

(3) Find the greatest angle in a triangle whose sides are 5, 6, and 7 ft. respectively, given that

$\log 6 = \cdot 7781513$, $\quad L \cos 39° 14' = 9 \cdot 8890644$,

and diff. for 15″ = ·0000258.

(4) Prove that if $A + B + C = 180°$

$$\tan\left(85° - \frac{5\,A}{12}\right) + \tan\left(85° - \frac{5\,B}{12}\right) + \tan\left(85° - \frac{5\,C}{12}\right)$$
$$= \tan\left(85° - \frac{5\,A}{12}\right) \tan\left(85° - \frac{5\,B}{12}\right) \tan\left(85° - \frac{5\,C}{12}\right).$$

(5) In the side $B\,C$ of a triangle a point D is taken so that $B\,D = m\,a$; prove that $A\,D^2 = c^2 + m^2\,a^2 - (a^2 - b^2 + c^2)\,m$.

(6) The angular elevation of a steeple at a place due south of it is 45°, and at another due west of the former station and at a distance a from it is 150°. Show that if the height of the steeple is

$$200\,\frac{\sqrt{3}}{3}\text{ ft.,}$$

then
$$a = 200\,(3^{\frac{1}{2}} + 3^{-\frac{1}{2}})\text{ ft.}$$

(7) Find the values of $\sin 900°$, $\sin \frac{35\,\pi}{12}$, $\cot 7125°$.

(8) Find the tangents of the greatest and least angles of the triangle whose sides are 13, 14, 15.

(9) Solve the equations—

1. $\cot x\,(\operatorname{cosec} 2\,x - \sin 2\,x) = 1.$
2. $\sin^{-1}\sqrt{x} + \sin^{-1}(1 - \sqrt{x}) = \cos^{-1}\sqrt{x}.$
3. $\sin 5\,\theta = 16\,\sin^5\theta.$

(10) Prove that

$$\tan 4\,\theta = \frac{2\tan\theta + \tan 2\,\theta - \tan^2\theta\tan 2\,\theta}{1 - \tan^2\theta - 2\tan\theta\tan 2\,\theta}.$$

Exercise 15.

(1) The interior angles of a polygon are in Arithmetical Progression: the least angle is 120° and the common difference 5°; find the number of sides.

(2) Show that—

1. $\operatorname{cosec} 4\,A = \cot 4\,A + \tan 2\,A.$
2. $\sqrt{\operatorname{cosec}^2 x - 1}\,\sqrt{2\operatorname{vers} x - \operatorname{vers}^2 x} = \cos x.$
3. $\cos 72° + \cos 60° = \cos 36°.$

(3) Solve the equations—

1. $\cos 8\,\theta + 2 = 3 \cos 4\,\theta$.

2. $(\sqrt{5} - 1) \cos 3\,x = 2 \sin^2 x + \cos 2\,x$.

3. $\tan a\,x = \cot b\,x$.

(4) If a regular octagon be described about a circle, and a similar octagon be inscribed in the circle, the area of the described figure is to that of the inscribed one as $4 - 2\sqrt{2} : 1$.

(5) In a triangle given $A = 53°\ 24'$, $B = 66°\ 27'$, $c = 338 \cdot 65$ yards, find the length of a.

Given

$L \sin 53°\ 24' = 9 \cdot 9046168$,	$\log 3 \cdot 3865 = \cdot 5297511$,
$L \cos 29°\ 51' = 9 \cdot 9381851$,	$\log 31346 = 4 \cdot 4961821$,
	diff. for $1 = \cdot 0000139$.

(6) Show that—

1. $$\tan^{-1} \frac{m^2 + 2\,m - 1}{m^2 + 2\,m} + \tan^{-1} \frac{1}{2\,m^2 + 4\,m - 1} = n\,\pi + \frac{\pi}{4}.$$

2. If $$\sin^{-1} \frac{\alpha}{\sqrt{1 + \alpha^2}} + \sin^{-1} \frac{\beta}{\sqrt{1 + \beta^2}} + \sin^{-1} \frac{\gamma}{\sqrt{1 + \gamma^2}} = \pi;$$

then $$\alpha\,\beta\,\gamma = \alpha + \beta + \gamma.$$

(7) In a triangle, if

$$\tan (A - B) = \frac{a^2 - b^2}{2\,a\,b}$$

prove that $A = B$, or $2 \operatorname{cosec} C = \cot A \cot B + 1$.

(8) Find the radius of a globe such that the distance measured upon its surface between two places on the same meridian whose latitudes differ by $1°\ 10'$ may be an inch.

(9) A building on a square base has two sides A B, C D parallel to the bank of a river. An observer on the bank in the same straight line as D A finds that A B subtends 45° at his eye. Having walked a yards along the bank he finds that D A subtends an angle $\sin^{-1} \frac{1}{3}$. Show that the length of each side of the building is $\frac{a}{\sqrt{2}}$ yards.

(10) If A B C, D E F be two triangles with the same perimeter, prove that the radius of the escribed circle touching

externally the side B C of the first is to the escribed circle touching externally E F of the second as $\tan \frac{A}{2} : \tan \frac{D}{2}$.

Exercise 16.

(1) Three angles whose sum is $\frac{3\pi}{4}$ are in Arithmetical Progression; show that the product of their tangents is unity.

(2) Show that the roots of the equation $4x^2 - 2\sqrt{5}\,x + 1 = 0$ are $\sin 18^\circ$ and $\cos 36^\circ$.

(3) In a triangle if $a = 352{\cdot}25$, $b = 513{\cdot}27$, $c = 482{\cdot}68$, find A, having given

$\log 6{\cdot}741 = {\cdot}8287243$,	
$\log 3{\cdot}2185 = {\cdot}5076535$,	
$\log 1{\cdot}6083 = {\cdot}2063401$,	$L \tan 20^\circ 38' = 9{\cdot}5758104$,
$\log 1{\cdot}9142 = {\cdot}2819873$,	$L \tan 20^\circ 39' = 9{\cdot}5761934$.

(4) Prove that

1. $$\frac{1 + \sin\theta}{1 - \sin\theta} = \frac{\sin^2\left(\frac{\pi}{4} + \frac{\theta}{2}\right)}{\sin^2\left(\frac{\pi}{4} - \frac{\theta}{2}\right)}.$$

2. $\operatorname{cosec} 2(A + B) + \operatorname{cosec} 2(A - B) - \cot 2(A + B) - \cot 2(A - B) = \sec(A + B)\sec(A - B)\sin 2A$.

(5) If $\tan x \cot y = \cos \alpha$, then

$$\tan(y - x) = \frac{\tan^2 \frac{\alpha}{2} \sin 2y}{1 + \tan^2 \frac{\alpha}{2} \cos 2y}.$$

(6) Solve the equations—

1. $$\left.\begin{aligned} \tan x + \tan y &= 4. \\ \tan 2x + \tan 2y &= 0. \end{aligned}\right\}$$

2. $$1 + \sin 2\theta = \tan\left(\frac{\pi}{4} + \theta\right).$$

(7) Show that

$$\tan^{-1}\frac{2-\sqrt{3}}{3}+\tan^{-1}\frac{2\sqrt{3}-1}{\sqrt{3}}=\frac{\pi}{3}.$$

(8) In any triangle prove that

$$\cos^2 2A+\cos^2 2B+\cos^2 2C-2\cos 2A\,.\cos 2B\,.\cos 2C=1.$$

(9) If D, E, F be the points where the inscribed circle touches the sides BC, CA, AB of the triangle ABC respectively, and if O_1 O_2 O_3 be the centres of the escribed circles touching the sides in this order, show that each of the triangles AEO_1, CFO_2, CDO_3 is equal to one half of the triangle ABC.

(10) Show that the area of a triangle ABC

$$=R^2(\sin^2 A+\sin^2 B-\sin^2 C)\cot\frac{A+B-C}{2}.$$

Exercise 17.

(1) In a triangle ABC, $A = 15°$ and $B = \frac{8}{3}C$, show that $\frac{b}{c}=\frac{\sqrt{3}}{\sqrt{2}}$.

(2) Solve the equations

1. $\tan(x+45°)+\tan(x-45°)=2\tan 60°.$
2. $\sin(x-a)=\sin x-\sin a.$

(3) In any triangle calculate cos A, cos B, cos C from $a = b\cos C + c\cos B$ and the two similar equations; also deduce the relations $\frac{\sin A}{a}=\frac{\sin B}{b}=\frac{\sin C}{c}$.

(4) Prove that

1. $4\sin A\sin\left(\frac{\pi}{3}-A\right)\sin\left(\frac{\pi}{3}+A\right)=\sin 3A.$

2. $\frac{\tan^2 2A-\tan^2 A}{1-\tan^2 2A\tan^2 A}=\tan A\tan 3A.$

3. $\cos\frac{\pi}{3}+\cos\frac{2\pi}{7}+\cos\frac{4\pi}{7}+\cos\frac{6\pi}{7}=0.$

(5) Eliminate x between the equations

$$\left.\begin{aligned}\operatorname{cosec} x-\sin x&=a\\ \sec x-\cos x&=b\end{aligned}\right\}$$

(6) If $\sin\theta = \dfrac{\sin\alpha + \sin\beta}{1 + \sin\alpha\sin\beta}$, find $\tan\left(\dfrac{\pi}{4} - \dfrac{\theta}{2}\right)$.

(7) Show that the area of a triangle

$$= \frac{a^2}{4}\sin 2(A + C) + \frac{b^2}{4}\sin 2(B + C).$$

(8) If O be the point of intersection of the perpendiculars drawn from the points A B C in any triangle on the opposite side, prove that $\dfrac{OA}{\cos A} = \dfrac{OB}{\cos B} = \dfrac{OC}{\cos C} = 2R$.

(9) Two towers whose heights are 180 ft. and 80 ft. stand on a horizontal plane. From the foot of each the angle of elevation of the other is taken and one angle is found to be double the other. Prove that the towers are 240 ft. apart, and that the greater angle of elevation is $\sin^{-1}\dfrac{3}{5}$.

(10) Show that in any triangle

$$\log a = \log(b - c) + L\cos\frac{A}{2} - L\cos\phi, \text{ where}$$

$$\tan\phi = \frac{b + c}{b - c}\tan\frac{A}{2}.$$

Exercise 18.

(1) Prove that

1. $\tan(\alpha + \beta) = \dfrac{\sin^2\alpha - \sin^2\beta}{\sin\alpha\cos\alpha - \sin\beta\cos\beta}$.
2. $\dfrac{1}{2}\left(1 + \tan\dfrac{A}{2}\right)^2 = \dfrac{1 + \sin A}{1 + \cos A}$.
3. $\{\cos(x + y) + \cos(x - y)\}\{\sin(x + y) - \sin(x - y)\}$
   $= 2\sin 2y\cos^2 x$.

(2) In any triangle prove

1. $b(\cot A + \cot B) = 2R\operatorname{cosec} A\sin C$.
2. $(b^2 - c^2)\tan\dfrac{B + C - A}{2} + (c^2 - a^2)\tan\dfrac{C + A - B}{2}$
   $+ (a^2 - b^2)\tan\dfrac{A + B - C}{2} = 0$

(3) Solve the equations

1. $\sin^{-1} 2x\sqrt{1 - x^2} = \sin^{-1}\alpha + \sin^{-1}\beta$.
2. $\tan^{-1}(x + 1) = 3\tan^{-1}(x - 1)$.

(4) Telegraph posts 70 yds. apart follow the direction of a straight road. A man standing opposite to one post finds that the angle subtended at his eye by the distance between the fourth and fifth posts is $\tan^{-1}\frac{1}{8}$. Find his distance from the line of posts.

(5) Show that

$$2\cos A = \sqrt{\{2 + \sqrt{(2 + \sqrt{(2 + \ldots + \sqrt{2 + 2\cos 2^n A}))}\}}.$$

(6) If $\alpha\ \beta$ are values of θ which satisfy the equation

$$\frac{\cos\theta}{a} + \frac{\sin\theta}{b} = \frac{1}{c}, \text{ then}$$

$$a\cos\frac{\alpha+\beta}{2} = b\sin\frac{\alpha+\beta}{2} = c\cos\frac{\alpha-\beta}{2}.$$

(7) Show that in any triangle

1. $\dfrac{\sin(A-B)}{\sin(B-C)} = \dfrac{a^2-b^2}{b^2-c^2}\cdot\dfrac{\sin A}{\sin C}$.

2. $2R\sin C = b\cos A + \sqrt{(a^2 - b^2\sin^2 A)}$.

(8) Find the relations between the sides of a triangle A B C,

1. When $\sin B\sin C = \cos^2\frac{A}{2}$.

2. $\sin B\sin C = \sin^2\frac{A}{2}$.

(9) Given $\cos(\alpha-\beta)\cos\alpha = a\sin\beta\cos\beta$, and

$$\sin(\alpha-\beta)\cos\alpha = \frac{a}{2}\cos^2\beta, \text{ prove that}$$

$$a = \frac{2\sin\beta}{1+3\sin^2\beta}.$$

(10) Prove that the distances from the centre of the inscribed to the centres of the escribed circles are

$$a\sec\frac{A}{2},\ b\sec\frac{B}{2},\ c\sec\frac{C}{2}, \text{ respectively.}$$

Exercise 19.

(1) Eliminate θ between the equations

$$\sin\theta + \cos\theta = a, \quad \cos 2\theta = b^2.$$

(2) Prove that $\cos^2 66^\circ - \cos^2 84^\circ = \dfrac{\sqrt{5}-1}{8}$.

(3) Solve the equations

1. $\cot\theta - \cot 3\theta = \cot 2\theta - \cot 4\theta$.
2. $\sin x - \sin 4x + \sin 7x = 0$.
3. $\sin 2x = \sqrt{2}\sin 3x$.

(4) In any triangle if $\cos A : \cos B : \cos C :: 25 : 19 : 7$, prove that $\sin A : \sin B : \sin C :: 5 : 6 : 7$.

(5) ABC is a triangle, $a = \sqrt{3}+1$, $b = \sqrt{3}-1$, $C = 60^\circ$; solve it.

(6) Prove that

$$\tan^{-1}\frac{2}{9} + \tan^{-1}\frac{2}{25} + \tan^{-1}\frac{2}{49} + \tan^{-1}\frac{2}{81} = \tan^{-1}\frac{1}{2} - \tan^{-1}\frac{1}{10}.$$

(7) If the cosines of the angles $\beta+\gamma$, $\alpha+\gamma$, $\beta-\gamma$ are in Geometrical Progression, then the sines of the angles $\alpha+\gamma-\beta$, γ, $\alpha+\gamma+\beta$ are also in Geometrical Progression.

(8) The side BC of a triangle ABC is bisected in D and AD, produced to meet the circumscribing circle of the triangle in E. Show that $2\,AD\,.\,AE = a^2 + 2bc\cos A$.

(9) I being the centre of the inscribed circle of a triangle, I_1, I_2, I_3 the centres of the escribed circles, show that

$$II_1\,.\,II_2\,.\,II_3 = \frac{IA^2\,.\,IB^2\,.\,IC^2}{r^3}.$$

(10) If $a = 30$, $b = 10$, $C = 53^\circ\,7'\,48''$, find c without finding A and B, using

$\log 2 = \cdot 30103$, $\quad L\cos 26^\circ\,33'\,54'' = 9\cdot 9515452$,
$\log 25298 = 4\cdot 4030862$, $\quad L\tan 26^\circ\,33'\,54'' = 9\cdot 6989700$,
$\log 25299 = 4\cdot 4031034$.

Exercise 20.

(1) In a triangle, if $\frac{A}{1} = \frac{B}{2} = \frac{C}{7}$, show that $\frac{a}{c} = \frac{\sqrt{5} - 1}{\sqrt{5} + 1}$.

(2) Solve the equations—

1. $$\sin x = 3 \sin y,$$ $$\sin x \cos y = \sin y\,(2 \cos y + 7 \sin y).$$

2. $$\tan^{-1} x - \tan^{-1} y = \cot^{-1} 2\,y - \cot^{-1} 2\,x = \frac{\pi}{4}.$$

(3) If $\sqrt{a^2 - x^2} \cos \beta + a \sin a = x \sin \beta$, find x.

(4) If l, m, n, be the lengths of the perpendiculars from the vertices of a triangle on the opposite sides, prove that the area of the triangle is $\frac{1}{2}\, l^{\frac{2}{3}}\, m^{\frac{2}{3}}\, n^{\frac{2}{3}}\, (\text{cosec A cosec B cosec C})^{\frac{1}{3}}$.

(5) An object 10 in. high, standing on a table, has a shadow 8 in. long cast by a lamp standing on the same table. When the lamp is raised 6 in. the shadow is shortened to 5 in. Show that the height of the lamp is 1 ft. 8 in.

(6) Show that in any triangle

$$\frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c} = \frac{1}{4\,R}\left(\frac{\sin^2 A + \sin^2 B + \sin^2 C}{\sin A \sin B \sin C}\right).$$

(7) If $\tan a$, $\tan \beta$, $\tan \gamma$, are in Arithmetical Progression, then will $\sin(\beta + \gamma - a)$, $\sin(\gamma + a - \beta)$, $\sin(a + \beta - \gamma)$, be also in Arithmetical Progression.

(8) Prove that

$$\sin 2\,a \sin \beta - \sin 2\,\beta \sin a = \big(\sin a - \sin \beta + \sin(\beta - a)\big)\big(\sin \beta + \sin a + \sin(\beta + a)\big)$$

(9) If $\sin A$, $\sin B$, $\sin C$ are in Harmonical Progression, so are $\sin^2 \frac{A}{2}$, $\sin^2 \frac{B}{2}$, $\sin^2 \frac{C}{2}$, A, B, C being angles of a triangle.

(10) If A D, B E, C F be the perpendiculars from the vertices on the opposite sides of the triangle A B C, prove that

$$\frac{1}{AD} + \frac{1}{BE} + \frac{1}{CF} = \frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c} = \frac{1}{r}.$$

EXAMINATION PAPERS.

Cambridge previous Examination, 1890.

Questions in Trigonometry set in the Mechanics Papers.

I. (A)

1. Trace the changes in the sign and magnitude of the *sine* of an angle, as the angle increases from 0° to 360°.

Find the angles in the second quadrant whose sines are the same as those of 833° and 1235° respectively.

2. Express the *tangent* of an angle in terms of its *sine*.

Prove the relation

$$2 \sec^2 A = 1 + \sec^4 A - \tan^4 A.$$

II. (A)

1. Define the cosecant of an angle, and prove the formula

$$\operatorname{cosec}^2 A = \cot^2 A + 1.$$

If the slope of a hill be a rise of 1 in 25, find to three decimal places, the secant, cosine and cotangent of its inclination to the horizon.

2. Prove that in any triangle A B C

$$AC^2 = AB^2 + BC^2 - 2\,AB \,.\, BC \cos ABC.$$

In a given triangle two sides are respectively 10 feet and 17 feet long, and the angle contained by them is 30°; find the length of the remaining side.

II. (B)

1. Define the sine of an angle, and prove the formula

$$\sin^2 A + \cos^2 A = 1.$$

If the slope of a hill be a rise of 1 in 20, find to three decimal places, the secant, the cosecant, and the tangent of its inclination to the horizon.

2. Prove that in any triangle the sides are proportional to the sines of the angles opposite to them.

In a given triangle one side is 4 feet long, the angle opposite to it is 45°, and one of the angles adjacent to it is 30°, find the smallest side.

Admission to the R. M. College, Sandhurst.

I.—June, 1890.

1. Define the sine, cosine, and tangent of an angle, and give the values of the following:

$$\sin 960°, \text{ cosec } (-510°), \tan 570°.$$

2. Prove *geometrically* that

$\sin (A - B) = \sin A \cos B - \cos A \sin B$ when $A - B$ and B are acute, but A is obtuse.

Prove that

$$\sin A \sin (B - C) + \sin B \sin (C - A) + \sin C \sin (A - B) = 0.$$

3. Prove that

$$\text{(i)} \quad \text{cosec}\, 2\theta + \cot 2\theta = \cot \theta;$$

$$\text{(ii)} \quad \tan \theta = \frac{\sin \theta + \sin 2\theta}{1 + \cos \theta + \cos 2\theta};$$

$$\text{(iii)} \quad \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{8} = \frac{\pi}{4}.$$

4. Express the cosine of an angle of a triangle, and the cosine of half that angle, in terms of the sides of the triangle. Show how to express the area in terms of the sides.

5. If $\tan \phi = \dfrac{a - b}{a + b} \cot \frac{1}{2} C$, prove that $c = (a + b)\dfrac{\sin \frac{1}{2} C}{\cos \phi}$ where a, b, c; A, B, C, denote the corresponding sides and angles of any triangle.

6. The sides of a triangle are 237 and 158, and the contained angle 66° 40′; use the formulæ given in the last question to find the base.

$$\begin{aligned}
\log 2 &= \cdot 30103 \\
\log 79 &= 1\cdot 89763 \\
\log 22687 &= 4\cdot 35578 \\
L \cot 33° \, 20' &= 10\cdot 18197 \\
L \sin 33° \, 20' &= 9\cdot 73998 \\
L \tan 16° \, 54' &= 9\cdot 48262 \\
L \tan 16° \, 55' &= 9\cdot 48308 \\
L \sec 16° \, 54' &= 10\cdot 01917 \\
L \sec 16° \, 55' &= 10\cdot 01921.
\end{aligned}$$

II.—December, 1890.

1. Give all the positive angles less than 360°, which satisfy the condition

$$\sin 2A = \sqrt{3} \cos 2A.$$

2. Simplify the expression

$2 \sec^2 A - \sec^4 A - 2 \operatorname{cosec}^2 A + \operatorname{cosec}^4 A$, giving the result in terms of tan A.

3. Prove the following identities:

(i) $\cos 3\theta = \cos\theta\,(2\cos 2\theta - 1)$;

(ii) $\sin^{-1}\dfrac{3}{\sqrt{73}} + \cos^{-1}\dfrac{11}{\sqrt{146}} + \sin^{-1}\dfrac{1}{2} = \dfrac{5\pi}{12}$.

4. If $A + B + C = 180°$, prove that

$$\cos\frac{A}{2}\cos\frac{B-C}{2} + \cos\frac{B}{2}\cos\frac{C-A}{2} + \cos\frac{C}{2}\cos\frac{A-B}{2} = \sin A + \sin B + \sin C.$$

5. Find expressions, in terms of the sides and area of a triangle, for the radii respectively of the circles inscribed in and circumscribed about the triangle. If x, y, and z, are the perpendiculars let fall from the angular points of the triangle upon the opposite sides a, b, and c, show that $\dfrac{bx}{c} + \dfrac{cy}{a} + \dfrac{az}{b} = \dfrac{a^2 + b^2 + c^2}{2R}$, where R is the radius of the circumscribing circle.

6. If $a = 1020$, $B = 107° 18'$, $C = 27° 10'$, find b; having given

$$\begin{aligned} L \sin 45° 32' &= 9{\cdot}8534902 \\ L \sin 72° 42' &= 9{\cdot}9798946 \\ \log 17 &= 1{\cdot}2304489 \\ \log 30 &= 1{\cdot}4771213 \\ \log 13645 &= 4{\cdot}1349735 \\ \log 13646 &= 4{\cdot}1350054. \end{aligned}$$

Admission to R. M. Academy, Woolwich.

I.—June, 1890.

1. Explain the various ways of measuring angles.

Calculate, in degrees, minutes and seconds, the angle whose circular measure is unity.

2. Prove that

$$\begin{aligned}&(1)\ \tan 60^\circ = \sqrt{3}.\\&(2)\ \cos 2A = 1 - 2\sin^2 A.\end{aligned}$$

3. Prove that

$$2\cos A = \pm\sqrt{1+\sin 2A} \pm \sqrt{1-\sin 2A}.$$

Determine the signs in the case when A is a negative angle between -45° and -90°.

4. Express sin A in terms of $\sin\frac{A}{3}$.

Why is it natural to expect that the result will be of the third degree in $\sin\frac{A}{3}$?

5. In any triangle prove the following relations:—

(1) $\cot A + \cot B = \frac{c}{b}\operatorname{cosec} A.$

(2) $\frac{\cos A}{\sin B \sin C} + \frac{\cos B}{\sin C \sin A} + \frac{\cos C}{\sin A \sin B} = 2.$

(3) $\cos(B-C)\sin 3A + \cos(C-A)\sin 3B$
$+ \cos(A-B)\sin 3C = 0.$

6. Explain the inverse notation of trigonometrical functions.
If $\operatorname{cosec}^{-1} x = \operatorname{cosec}^{-1} a + \operatorname{cosec}^{-1} b$, find the value of x in terms of a and b.

7. Show how to solve a triangle when two angles and a side opposite to the third angle are given.
From the top of a hill the angles of depression of two successive milestones on level ground, and in the same vertical plane as the observer, are found to be 5° and 10° respectively. Find the height of the hill and the horizontal distance to the nearest milestone.

8. Write down formulæ adapted to the use of logarithms for solving a triangle when two sides and the included angle are given.
Solve the triangle in which

$$a = 242\cdot 5,\ b = 164\cdot 3,\ C = 54^\circ\ 36'.$$

II.—December, 1890.

1. Prove that $\sin 30^\circ = \frac{1}{2}$, and find the values of sin 210° and sin 750°.

2. Taking $A + B$ less than 90°, prove by a geometrical figure that

$$\sin (A + B) = \sin A \cos B + \cos A \sin B.$$

Prove that $$4 \sin^2 75^\circ = 2 + \sqrt{3}.$$

3. Find $\tan A$ in terms of $\tan 2A$, and show, *à priori*, that two values will be obtained.

Prove that $$\tan 7^\circ 30' = \sqrt{6} + \sqrt{2} - \sqrt{3} - 2.$$

4. Solve the equation

$$\sin 11\theta + \sin 5\theta = \sin 8\theta.$$

5. Prove that, for a triangle ABC,

$$(1)\quad a = b \cos C + c \cos B.$$

$$(2)\quad \frac{b^2 - c^2}{\tan A} + \frac{c^2 - a^2}{\tan B} + \frac{a^2 - b^2}{\tan C} - 0.$$

6. Prove that the distances of the centre of the inscribed circle of a triangle ABC from the centres of the escribed circles are respectively

$$a \sec \frac{A}{2},\ b \sec \frac{B}{2},\ c \sec \frac{C}{2}.$$

7. Solve the equation

$$\tan^{-1}(x + 1) + \tan^{-1}(x - 1) = \tan^{-1} \frac{8}{31};$$

and prove that

$$\sec^2 (\tan^{-1} 2) + \operatorname{cosec}^2 (\cot^{-1} 3) = 15.$$

8. Prove that, in a triangle ABC,

$$\tan \frac{A}{2} = \left\{ \frac{(s - b)(s - c)}{s(s - a)} \right\}^{\frac{1}{2}}.$$

If $a = 21$, $b = 23$, and $c = 32$, calculate the angle A.

9. Looking out of a window, with his eye at the height of 15 feet above the roadway, an observer finds that the angle of elevation of the top of a telegraph post is $17^\circ 18' 35''$, and that the angle of depression of the foot of the post is $8^\circ 32' 15''$; calculate the height of the telegraph post, and its distance from the observer.

Cambridge Local Examinations, December, 1889.

Junior Candidates.

1. Define the tangent and the secant of an angle. Show from your definition that the difference of their squares is always the same.

2. What is the circular measure of an angle?

Find the number of degrees in the angle subtended at the centre of a circle whose radius is 7 inches by an arc 11 inches long, assuming $\pi = \dfrac{22}{7}$.

3. Determine directly by a geometrical construction the value of $\cos 120°$.

4. Show that, whatever be the magnitude of θ,

$$\cos(\theta + 90°) + \sin\theta = 0.$$

5. Give a geometrical proof of the formulæ

(i) $\cos(A - B) = \cos A \cos B + \sin A \sin B$;

(ii) $\cos A + \cos B = 2 \cos\frac{1}{2}(A + B) \cos\frac{1}{2}(A - B)$.

6. Prove that

(i) $2\operatorname{cosec} 4\theta + 2\cot 4\theta = \cot\theta - \tan\theta$;

(ii) $(\cos 2A + \cos 2B)^2 + (\sin 2A + \sin 2B)^2 = 4\cos^2(A - B)$.

7. Find all the values of x between 0 and 2π inclusive, which satisfy the equations

(i) $\cos^2 x + \sin x - 1 = 0$;

(ii) $\cos 5x + \cos 3x + \cos x = 0$.

8. Show that the sides of a plane triangle ABC are proportional to the sines of the opposite angles.

Prove that

$$a \sin(B - C) + b \sin(C - A) + c \sin(A - B) = 0.$$

9. State (without proof) in what cases three of the six parts of a triangle determine it without ambiguity.

Illustrate the ambiguous case geometrically.

10. The lengths of the two sides of a right-angled triangle are respectively 9·65 inches and 12·24 inches. Find the angles, having given

$\log 2 = \cdot 30103$,	$L \tan 38° 15' = 9\cdot 89671$,
$\log 153 = 2\cdot 18469$,	$L \tan 38° 16' = 9\cdot 89697$.
$\log 193 = 2\cdot 28556$,	

OXFORD LOCAL EXAMINATIONS, JULY, 1890.

Junior Candidates.

1. The circular measure of one angle of a triangle is $2\frac{3}{4}$, and another angle is $19° 26' 40''$. Express the remaining angle in degrees, minutes, and seconds. (Assume $\pi = 3\frac{1}{7}$).

2. Find a formula for all the angles which satisfy the equation $\cot 2x = \frac{1}{\sqrt{3}}$.

Explain why this equation gives two values of $\cot x$.

3. Show that

(1) $\cos(A - B) = \cos A \cos B + \sin A \sin B$;

(2) $\tan A + \frac{1}{2}\cos 2A \sec A \operatorname{cosec} A = \operatorname{cosec} 2A$;

(3) $\sec 15° = \sqrt{2}(\sqrt{3} - 1)$;

(4) $\sin(A + B) + \sin(B + C) + \sin(C + A)$

$= 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$, where A, B, C are the angles of a triangle.

4. In a plane triangle of which A, B, C are the angles a, b, c the sides, and $s = \frac{a+b+c}{2}$, show that

$$(1)\ \tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}};$$

$$(2)\ \tan\frac{A-B}{2} = \frac{a-b}{a+b}\cot\frac{C}{2}.$$

5. Given that $\log 2 = \cdot 3010300$, $\log 3 = \cdot 4771213$, find the logarithms of 225, and $\cdot 00\dot{3}$, and the number of digits in 2^{50}.

6. In a plane triangle of which A, B, C, are the angles, a, b, c, the sides, S, the area, R, r, the radii of the circumscribed and inscribed circles, prove that

$$(1)\ r = \frac{2S}{a+b+c};$$

$$(2)\ Rr\left(\frac{1}{bc} + \frac{1}{ca} + \frac{1}{ab}\right) = 1.$$

7. Find the angles A and B of a triangle A, B, C, having given $a = 17$, $b = 6$, $C = 127° 40'$:

$$\begin{aligned} \log 11 &= 1\cdot0413927, \\ \log 23 &= 1\cdot3617278, \\ L \cot 63° 50' &= 9\cdot6913809, \\ L \tan 13° 13' &= 9\cdot3707994, \\ L \tan 13° 14' &= 9\cdot3713667. \end{aligned}$$

8. A man observes the angle of elevation of the top of a hill to be $\alpha°$. After walking 500 yards in a horizontal plane towards the base of the hill, he finds the angle of elevation to be 60°. Find the height of the hill, being given that $\cot^2 \alpha° = 1\frac{1}{3}$.

ANSWERS.

PART I.

I (*a*) p. 2.

1. $60°$; $66^{g}\,66^{`}\,66\cdot\dot{6}^{``}$.
2. $16°\,40'$; $18^{g}\,51^{`}\,85\cdot\dot{1}8\dot{5}^{``}$.
3. $61°\,52'\,30''$; $68^{g}\,75^{`}$.
4. $8'\,38\cdot4''$; $16^{`}$.
5. $33°\,45'$; $37^{g}\,50^{`}$.
6. $83°\,48'\,45''$; $93^{g}\,12^{`}\,50^{``}$.
7. $103°\,7'\,30''$; $114^{g}\,58^{`}\,33\cdot\dot{3}^{``}$.
8. $214°\,51'\,9''$; $238^{g}\,72^{`}\,50^{``}$.
9. $\cdot15$.
10. $\cdot50625$.
11. $\cdot064$.
12. $\cdot19008\dot{3}$.
13. $\cdot275\dot{1}$.
14. $\cdot725$.
15. $\cdot121724$.
16. $\cdot021824$.
17. $\cdot73074$.
18. $\cdot033004$.
19. $\cdot0327$.
20. $\cdot000017$.
21. $34^{g}\,52^{`}\,77\cdot\dot{7}^{``}$; $2^{g}\,10^{`}\,4^{``}$; $20^{g}\,70^{`}\,70\cdot\dot{7}\dot{0}^{``}$; $25^{g}\,38^{`}\,53\cdot\dot{8}5\dot{3}^{``}$.

I (*b*) p. 4.

1. $32^{g}\,32^{`}\,50^{``}$.
2. $87^{g}\,35^{`}$.
3. $12^{g}\,6^{`}\,25^{``}$.
4. $73^{g}\,84^{`}\,62\cdot\dot{9}6\dot{2}^{``}$.
5. $37^{g}\,1^{`}\,66\cdot\dot{6}^{``}$.
6. $30^{g}\,56^{`}\,94\cdot\dot{4}^{``}$.
7. $15°\,51'\,45''$.
8. $9°\,0'\,40\cdot5''$.
9. $48°\,41'\,38\cdot256''$.
10. $28°\,2'\,27\cdot06''$.
11. $37°\,49'\,38\cdot82''$.
12. $6°\,28'\,50\cdot592''$.
13. $120°$, $133^{g}\,33^{`}\,33\cdot\dot{3}^{``}$.
14. $56°\,48'\,45''$.
15. $91^{g}\,66^{`}\,66\cdot\dot{6}^{``}$.
16. $36°$, $72°$; 40^{g}, 80^{g}.
17. $30°$, $120°$; $33\cdot\dot{3}^{g}$, $133\cdot\dot{3}^{g}$.
18. $30°$, $60°$. $90°$; $33\cdot\dot{3}^{g}$, $66\cdot\dot{6}^{g}$, 100^{g}.
19. $48°\,50'\,15\cdot36''$; $54^{g}\,26^{`}\,40^{``}$.
20. $39°$, $60°$, $81°$; $43\cdot\dot{3}^{g}$, $66\cdot\dot{6}^{g}$, 90^{g}.

II (*a*) p. 7.

1. 7 ft.
2. 280.
3. 5 mi.
4. 65476 mi. nearly.
5. 2 ft.
6. $69\frac{2}{3}$ mi.

II (*b*) p. 9.

1. 12 ft. 2. 2·1 in. 3. 1·6. 4. 1·2.
5. $125\frac{4}{7}$ ft. 6. $\frac{5}{8}^c$. 7. 18 mi. per hr. 8. 28 ft. $10\frac{1}{2}$ in.

II (*c*) p. 12.

1. $\frac{2\pi}{15}$. 2. $\frac{7\pi}{20}$. 3. $\frac{\pi}{40}$.
4. $\cdot 0345\,\pi$. 5. $\frac{125}{288}\pi$. 6. $\cdot 12\,\pi$.
7. $\cdot 25625\,\pi$. 8. $\cdot 078125\,\pi$. 9. $\cdot 1171875\,\pi$.
10. $\cdot 4003125\,\pi$. 11. $30°$; $33\cdot\dot{3}^g$. 12. $67\frac{1}{2}°$; 75^g.
13. $22\frac{1}{2}°$ 25^g. 14. $90°$, 100^g. 15. $47\frac{13}{16}°$, $53\frac{1}{8}^g$.
16. $22\frac{10}{11}°$; $25\frac{5}{11}^g$. 17. $71\frac{13}{11}°$, $79\frac{6}{11}^g$. 18. $9\cdot 6°$; $10\cdot\dot{6}^g$.
19. $94\frac{1}{2}°$, 105^g. 20. $\frac{180\,\theta}{\pi}$ $\frac{200\,\theta}{\pi}$. 21. $60°$, $30°$.
22. $75°$, $75°$, $30°$. 23. $71\frac{151}{819}°$. 24. $\frac{7^c}{22}$, or $18\frac{27}{121}°$.
25. $46\frac{2}{3}$. 26. $\frac{44^c}{2555}$.

II (*d*) p. 13.

1. 18″. 2. 53·69 in.
3. 4400 mi. 4. 8567000 mi.

III (*a*) p. 16.

1. $\sin A = \frac{4}{5}, \cos A = \frac{3}{5}, \tan A = \frac{4}{3}$, &c. 2. $-\frac{11}{3}$.

3. $\cos ACD = \frac{CD}{AC}, \tan BCD = \frac{BD}{CD}$;

$\sin ABC = \frac{CD}{BC} = \frac{AC}{AB}$;

$\cot BAC = \frac{AC}{BC} = \frac{AD}{CD}$.

4. $\frac{70}{13}, \frac{29}{26}$.

5. $\sin A = \frac{DE}{AD} = \frac{CD}{AC} = \frac{BC}{AB}$;

$\cos A = \frac{AE}{AD} = \frac{AD}{AC} = \frac{AC}{AB}$;

$\tan A = \frac{DE}{AE} = \frac{DC}{AD} = \frac{BC}{AC}$.

6. $\frac{3}{5}, \frac{4}{5}, \frac{5}{4}, \frac{3}{4}, \frac{1}{5}$.

7. sines $\frac{12}{13}, \frac{5}{13}$; cosines $\frac{5}{13}, \frac{12}{13}$; tangents $\frac{12}{5}, \frac{5}{12}$; &c.

III (*b*) p. 20.

1. $\sin A = \dfrac{\tan A}{\sqrt{1+\tan^2 A}}$, $\cos A = \dfrac{1}{\sqrt{1+\tan^2 A}}$; $\cot A = \dfrac{1}{\tan A}$, &c.

2. $\cot A = \dfrac{\cos A}{\sqrt{1-\cos^2 A}}$.

3. $\cos A = \sqrt{1-\sin^2 A}$, $\tan A = \dfrac{\sin A}{\sqrt{1-\sin^2 A}}$, &c.

4. $\sin A = \dfrac{3}{5}$, $\cos A = \dfrac{4}{5}$.
5. $\sin A = \dfrac{20}{101}$, $\tan A = \dfrac{20}{99}$.

6. $\cos A = \dfrac{12}{13}$, $\tan A = \dfrac{5}{12}$.
7. $\sin A = \dfrac{33}{65}$, $\cos A = \dfrac{56}{65}$.

8. $\sin A = \dfrac{24}{25}$ $\cos A = \dfrac{7}{25}$.
10. $\dfrac{1343}{654}$.

IV (*a*) p. 26.

1. $\dfrac{3}{2}$
2. 1.
3. 2.
4. $\dfrac{1}{4}$.
5. 5.
6. $2-\sqrt{6}$.
7. 1.
8. 1.
9. $\dfrac{3\sqrt{3}}{1+2\sqrt{2}}$.
10. -1.

IV (*b*) p. 28.

1. $\dfrac{9}{4}$.
2. $\dfrac{4}{3}$.
3. 1.
4. $\dfrac{1}{2}$
5. ∞.
6. $2+\sqrt{3}$.
7. 1.
8. $\sqrt{2}-2$.
9. $\dfrac{4}{3}$.
10. $\dfrac{1}{\sqrt{2}}$.

V. p. 33.

1. 30°.
2. 45°.
3. 45°.
4. 90°.
5. 30°.
6. 0, 60°.
7. $22\frac{1}{2}$°.
8. 45°.
9. 0, 90°.
10. 30°.
11. 0, 60°.
12. 90°, $\cot^{-1} 2$.
13. 30°, or 60°.
14. 45°, 60°.
15. 30°, 45°.
16. $\tan^{-1}\dfrac{1}{2}$, $\tan^{-1}\dfrac{1}{4}$.
17. 15°.
18. 8°.
19. 21°, 6°.
20. 45°, 60°.
21. $A = 60°, 0$; $B = 0, 60°$
22. $A = 30°, 60°$; $B = 60°, 30°$.
23. $A = 45°$, $B = 30°$.

VI. p. 38.

1. $A = B = 45°$, $b = 50$.
2. $A = B = 45°$, $c = 8\cdot4$.
3. $B = 30°$, $a = 8\cdot65$, $c = 10$.
4. $B = 45°$, $a = 5$, $c = 7$.
5. $A = 60°$, $a = 25\cdot95$, $b = 15$.
6. $A = 60°$, $B = 30°$, $a = 25\cdot95$.
7. $A = 30°$, $a = 100$, $c = 200$.
8. $B = 45°$, $b = a = 10$.
9. $B = 60°$, $A = 30°$, $a = 20$.
10. $A=30°$, $B=60°$, $c=24$, $b=20\cdot76$.
11. $A = 30°$, $B = 60°$, $c = 20$, $b = 17\cdot3$.
12. $A = B = 45°$, $a = b = 20$.
13. $A = 60°$, $B = 30°$, $a = 17\cdot3$, $c = 20$.
14. $a = 20$, $b = 15$.
15. $b = 10$, $c = 26$.
16. $a = 49\cdot5$, $b = 50\cdot5$.
17. $a = 22$, $b = 120$.
18. $b = 2\cdot4$, $c = 7\cdot4$.

VII (*a*) p. 42.

1. 45°.
2. 173 ft.
3. 100 yds.
4. 101·6 ft.
5. 12 ft.
6. 173 yds.
7. 1920 ft. nearly.
8. 15 ft.; $13\frac{11}{13}$ ft.
9. 47·775 ft.
10. 642·4 yds.; 1117·6 yds., or 2402·4 yds., 4162·4 yds.
12. $\frac{10}{29}$ mi. = 606·9 yds.
13. $1\frac{1}{3}$ min.
14. 150 ft.
15. 23 ft.
16. 240 ft.
18. 2·97 mi.
19. 72 ft.
20. 6·92 mi.

VII (*b*) p. 46.

1. 3733 ft.
2. 106 ft.
3. 97·9 ft.
4. 1522 yds.
5. 45°.
6. 35° 44′.
7. 2694 ft.
8. 692 yds.

Miscellaneous Examples on Part I., p. 47.

1. 3·627 ft.
2. 10·8385 yds.
3. $83^{g}\,23'\,50''$.
4. ·8, ·75, ·4.
5. $52\frac{1}{2}°$.
6. 13° 57′; ·2434734.
8. ·5547; ·8320; ·1680.
10. 30°; 60°.
11. 1° 48′; 2^{g}.
13. $111\cdot\dot{1}^{g}$; $7494\cdot\dot{8}1\dot{4}'$.
14. 54°; 36°.
15. 8.
16. $68\frac{8}{11}°$.
17. 6160 ft.
18. 60°; 0°.
20. 26·6 ft.
21. 30°.
22. 40 ft.
24. 39° 22′ 30″.
25. 30°; 71° 15′; 78° 45′.
28. The sine = ·447, the cosine = ·894.
29. $\sin A = \frac{3}{5}$ or $\frac{1}{3}$; $\sin B = \frac{1}{3}$ or $\frac{3}{5}$.
30. 60.
31. $\frac{\sqrt{3}}{2}$.
32. 4·5112 mi.
33. 20° 45′ $22\frac{1}{2}$″.
34. $\frac{1+x}{\sqrt{2+2x^2}}$; $\frac{\sqrt{2+2x^2}}{1-x}$.
37. 60°.
38. $21\frac{1}{2}$ ft.
40. 341 ft.

PART II.

VIII (*a*) p. 54.

1. $\frac{\sqrt{3}-1}{2\sqrt{2}}$. 2. $2+\sqrt{3}$. 3. $2-\sqrt{3}$, $\sqrt{6}+\sqrt{2}$, $\sqrt{6}-\sqrt{2}$.

4. $\frac{33}{65}$. 5. $\frac{56}{65}$. 6. $\frac{56}{33}$.

7. $\frac{273}{305}$. 8. $\frac{3}{4}$. 9. $\frac{1525}{1363}$.

VIII (*b*) p. 57.

1. $\frac{\sqrt{3}+1}{2\sqrt{2}}$. 2. $2-\sqrt{3}$. 3. $2+\sqrt{3}$; $\sqrt{6}-\sqrt{2}$; $\sqrt{6}+\sqrt{2}$.

4. $\frac{33}{65}$. 5. $\frac{33}{56}$. 6. $\frac{4}{5}$. 7. $\frac{1525}{1363}$.

8. $\frac{24}{25}$. 9. $\frac{204}{253}$. 10. $\frac{793}{252}$.

VIII (*c*) p. 60.

3. $\sin A\cos B\cos C+\sin B\cos C\cos A-\sin C\cos A\cos B+\sin A\sin B\sin C$.

4. $\cos A\cos B\cos C+\sin A\sin B\cos C-\sin B\sin C\cos A+\sin C\sin A\cos B$.

5. $\frac{\cot A\cot B\cot C-\cot A-\cot B-\cot C}{\cot A\cot B+\cot B\cot C+\cot C\cot A-1}$.

6. $\frac{\tan A+\tan B-\tan C+\tan A\tan B\tan C}{1-\tan A\tan B+\tan B\tan C+\tan C\tan A}$.

IX (*a*) p. 65.

1. $\frac{24}{25}$; $\frac{7}{25}$; $\frac{24}{7}$. 2. $\frac{336}{625}$; $\frac{527}{625}$; $\frac{336}{527}$. 3. $\frac{720}{1681}$; $\frac{1519}{1681}$; $\frac{720}{1519}$.

4. $\frac{4\sqrt{5}}{9}$; $\frac{1}{9}$; $4\sqrt{5}$. 5. $\frac{4}{5}$; $\frac{3}{5}$; $\frac{4}{3}$. 6. $\frac{1320}{3721}$; $\frac{3479}{3721}$; $\frac{1320}{3479}$.

7. $\frac{\sqrt{2-\sqrt{2}}}{2}$; $\frac{\sqrt{2+\sqrt{2}}}{2}$; $\sqrt{2}-1$. 8. $\frac{1}{3}$; $\frac{2\sqrt{2}}{3}$; $\frac{1}{2\sqrt{2}}$.

9. $\frac{1}{\sqrt{5}}$; $\frac{2}{\sqrt{5}}$; $\frac{1}{2}$. 10. $\frac{1-m}{\sqrt{2(1+m^2)}}$; $\frac{1+m}{\sqrt{2(1+m^2)}}$; $\frac{1-m}{1+m}$.

IX (*b*) p. 68.

1. $\frac{7}{128}$. 2. $\frac{23}{27}$. 3. $\frac{11}{2}$. 4. $\frac{13}{9}$. 5. $\frac{7}{128}$.

IX (c) p. 70.

7. ·9945. 8. ·9781. 9. ·1583. 10. ·9135.

X. p. 72.

1. $\cos 3A + \cos A$.
2. $\sin 8A - \sin 2A$.
3. $\cos 2A - \cos 5A$.
4. $\sin(3A + 2B) + \sin(3A - 2B)$.
5. $\sin 7A + \sin 3A$.
6. $\frac{1}{2}(\cos 2A + \cos A)$.
7. $\frac{1}{2}(\sin 60^\circ - \sin 30^\circ)$.
8. $\frac{1}{2}(\sin 60^\circ + \sin 30^\circ)$.
9. $\sin 2x + \sin 6y$.
10. $\cos 90^\circ + \cos A = \cos A$.

XI (a) p. 75.

1. $\tan 3A$.
2. $\cot A$.
3. $\tan 2A$.
4. $\tan 2A$.
5. $\cot(A + 2B)$.
6. $\cot A$.
7. $\tan \frac{A}{2}$.
8. $\tan \frac{A}{4}$.
9. $\frac{\cos 4A}{\cos 6A}$.
10. $2 \sin A$.

XI. (b) p. 80.

1. $\frac{\pi}{2}, \frac{\pi}{9}$.
2. $45^\circ, 60^\circ$.
3. $0^\circ, 7\frac{1}{2}^\circ$.
4. $22\frac{1}{2}^\circ$; 45°; 90°.
5. $90^\circ, \frac{60^\circ}{n-1}$.
6. 0°.
7. $30^\circ + \frac{a}{3}$.
8. 0.
9. $0, \frac{\pi}{2(m+n+1)}$.
10. 12°.

XI (c) p. 81.

1. $2 \sin 45^\circ \cos(45^\circ - 2A)$.
2. $2 \sin 45^\circ \cos(45^\circ + A)$.
3. $2 \cos(45^\circ + B) \cos(45^\circ - A)$.
4. $2 \sin\left(90^\circ - \frac{A}{2}\right) \cos \frac{A}{2}$.
5. $2 \sin\left(45^\circ + \frac{3A}{2}\right) \cos\left(45^\circ - \frac{3A}{2}\right)$.
6. $4 \sin(A + 15^\circ) \cos(A - 15^\circ)$.
7. $4 \cos(A + 30^\circ) \cos(A - 30^\circ)$.
8. $2\sqrt{2} \sin\left(22\frac{1}{2}^\circ - \frac{A}{2}\right) \cos\left(22\frac{1}{2}^\circ + \frac{A}{2}\right)$.

XI (d) p. 82.

4. $A = 60^\circ$; $B = 30^\circ$.
5. $\cos 2B + \sin 2A$.

Miscellaneous Examples on Part II., p. 85.

13. $\frac{117}{25}$. **18.** $\frac{\sqrt{3}}{2}$. **22.** $0, \frac{\pi}{4}$.

23. $\frac{323}{325}$. **25.** $1 + b = a^2$. **27.** $\cos 2A + \sin 2B$. **31.** $30°$; $90°$.

62. $\sin A = \frac{\sqrt{7} \pm \sqrt{3}}{\sqrt{20}}$; $\sin B = \frac{\sqrt{3} \pm \sqrt{2}}{\sqrt{10}}$. **63.** $0°, 90°$.

67. $\frac{119}{65}$. **72.** $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$. **73.** $\frac{\pi}{2}$ or $\frac{\pi}{3(2n-1)}$.

95. $\frac{\pi}{4}$; $\frac{\pi}{42}$. **98.** 1; $\pm 2\sin^2\frac{\alpha}{2}$; $\pm 2\cos^2\frac{\alpha}{2}$.

PART III.

XII. p. 93.

1. $a + x$; $b - x - a$; $a - b$; $2b - x - a$; $2a + 2x - b$; $b - a$.
3. 2nd; 4th; 2nd; 3rd; 1st; 3rd; 3rd; 4th; 4th; 2nd; 2nd; 2nd.

XIII. p. 97.

1. $-$.	**2.** $+$.	**3.** $+$.	**4.** $-$.	**5.** $-$.
6. $-$.	**7.** $-$.	**8.** $-$.	**9.** $-$.	**10.** $+$.
11. $+$.	**12.** $+$.	**13.** $-$.	**14.** $+$.	**15.** $-$.
16. $+$.	**17.** $-$.	**18.** $-$.	**19.** $-$.	**20.** $-$.

XIV. p. 101.

7. $\cos x + \sin x = \sqrt{2}\sin\left(x + \frac{\pi}{4}\right)$. **9.** $\frac{\sin x}{1 + \cos x} = \tan\frac{x}{2}$.

XV (*a*) p. 106.

1. -1. **2.** $\frac{1}{\sqrt{3}}$. **3.** -2. **4.** -1. **5.** $-\frac{\sqrt{3}}{2}$.

6. $\frac{1}{2}$. **7.** $-\sqrt{2}$. **8.** $-\sqrt{3}$. **9.** $-\sqrt{3}$. **10.** $\frac{2-\sqrt{3}}{2}$.

11. 0. **12.** -1. **14.** $-\sin\theta$.

XV (*b*) p. 109.

4. $-\sin A$. **5.** $\cot A$. **6.** $-\cos A$.

7. $\tan A$. **8.** $\frac{-\sqrt{3}+1}{2}$. **9.** $-2 + \frac{5\sqrt{3}}{3}$.

XVI (a) p. 113.

1. $2n\pi \pm \frac{\pi}{3}$.
2. $n\pi + (-1)^n \frac{\pi}{3}$.
3. $n\pi + \frac{3\pi}{4}$.
4. $2n\pi \pm \frac{\pi}{4}$.
5. $(2n+1)\frac{\pi}{4}$.
6. $\frac{n\pi}{3} + \frac{\pi}{12}$.
7. $\frac{n\pi}{2} + (-1)^n \frac{7\pi}{12}$.
8. $\frac{n\pi}{4} + (-1)^n \frac{\pi}{12}$.
9. $\frac{(4n-1)\pi}{4} + (-1)^n \frac{\pi}{6}$.
10. $2n\pi$ or $2n\pi + \frac{\pi}{2}$.
11. $n\pi - \frac{\pi}{3}$.
12. $n\pi + \frac{\pi}{6}$
13. $\frac{n\pi}{2} + (-1)^n \frac{\pi}{6}$.
14. $2n\pi + a$.
15. $\frac{2n+1}{6}\pi - \frac{a}{3}$.
16. $n\pi + \frac{\pi}{4}$.

XVI (b) p. 115.

1. $n\pi + (-1)^n \frac{7\pi}{6}$.
2. $(2n+1)\frac{\pi}{6}, n\pi \pm \frac{\pi}{3}$.
3. $n\pi + \frac{\pi}{4}, n\pi + \frac{\pi}{6}$.
4. $2n\pi + \frac{\pi}{3}$.
5. $(2n+1)\pi, 2n\pi + \frac{\pi}{2}$.
6. $2n\pi, 2n\pi \pm \frac{\pi}{4}, 2n\pi \pm \cos^{-1}\left(\frac{-1}{2\sqrt{2}}\right)$.
7. $\frac{n\pi}{2} \pm \frac{\pi}{4}$.
8. $(2n+1)\pi, (2n+1)\frac{\pi}{2}, 2n\pi \pm \frac{\pi}{3}$.
9. $n\pi, (2n+1)\frac{\pi}{8}$.
10. $n\pi + a;\ \frac{n\pi}{2} - a$.
11. $2n\pi + \frac{\pi}{4}, 2n\pi + \frac{7\pi}{12}$.
12. $\frac{n\pi}{12}$.

XVIII. p. 126.

1. $\frac{\sqrt{3}-1}{2\sqrt{2}}$.
2. $-\frac{\sqrt{2+\sqrt{2}}}{2}$.
3. $\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2};\ -\frac{\sqrt{2-\sqrt{2+\sqrt{2}}}}{2}$.
4. $2\sin A = \sqrt{1+\sin 2A} + \sqrt{1-\sin 2A};\ 2\cos A = \sqrt{1+\sin 2A} - \sqrt{1-\sin 2A}$.
5. $2\sin A = -\sqrt{1+\sin 2A} + \sqrt{1-\sin 2A};\ 2\cos A = -\sqrt{1+\sin 2A} - \sqrt{1-\sin 2A}$.
6. Same signs as **5**.
7. $2\sin A = \sqrt{1+\sin 2A} - \sqrt{1-\sin 2A};\ 2\cos A = \sqrt{1+\sin 2A} + \sqrt{1-\sin 2A}$.
8. Same signs as **4**.
9. Same signs as **7**.

10. $2n\pi + \frac{\pi}{4}$ and $2n\pi + \frac{3\pi}{4}$. 11. $2n\pi + \frac{5\pi}{4}$ and $2n\pi + \frac{7\pi}{4}$.

12. $2n\pi + \frac{3\pi}{4}$ and $2n\pi + \frac{5\pi}{4}$.

Miscellaneous Examples, p. 127.

3. $\frac{1}{2}\{\sqrt{1+\sin A} + \sqrt{1-\sin A}\}$.

4. $2n\pi \pm \frac{2\pi}{3}$; $2n\pi \pm \frac{\pi}{5}$; $2n\pi \pm \frac{3\pi}{5}$.

5. $\frac{4ab(a^2-b^2)}{(a^2+b^2)^2}$; $-\left\{\frac{\sqrt{a^2+b^2}-a}{2\sqrt{a^2+b^2}}\right\}^{\frac{1}{2}}$.

11. $-\frac{897}{1025}$. 12. $n\pi + \frac{\pi}{8}$. 13. $\frac{\sqrt{5}+1}{2\sqrt{3}}$

14. $\frac{204}{253}$. 17. $n\pi + \frac{\pi}{4}$. 22. $n\pi + (-1)^n\frac{\pi}{3}$.

24. $\frac{1}{2}\{\sqrt{1+\sin A} + \sqrt{1-\sin A}\}$. 26. $-\frac{3}{5}$.

32. $(2n+1)\frac{\pi}{2}$, $(2n+1)\frac{\pi}{6}$. 34. $\frac{3}{5}$, $-\frac{4}{5}$.

PART IV.

Logarithms.—XIX (*a*) p. 132.

1. 1024. 2. 64. 3. 4096. 4. 32768.
5. 256. 6. 128. 7. £8192. 8. £273 1*s*. 4*d*.

XIX (*b*) p. 134.

1. 3, 5, 8, 9. 2. 2, 4, 6. 3. $\frac{3}{2}, 2, \frac{5}{2}, \frac{7}{2}, \frac{9}{2}$.

4. $\frac{2}{3}, \frac{8}{3}, 4, \frac{14}{3}, 6$. 5. $\frac{2}{3}, 2, \frac{8}{3}, \frac{10}{3}$. 6. 3.

7. 9. 8. $2\sqrt{2}$. 9. $\sqrt{5}$.
10. $2\sqrt{3}$. 11. 7. 12. $2\sqrt{2}$. 13. $3\sqrt[3]{3}$.

XIX (*c*) p. 138.

1. $\frac{3}{2}(\log a + \log b)$. 2. $a\log x + b\log y - c\log z$.

3. $\frac{1}{3}\log a + \frac{2}{3}\log x - \log y$.

4. $\frac{1}{2}\log a+\frac{3}{4}\log x-\frac{1}{2}\log y-\log b+\frac{1}{4}\log z$.

5. $3\log(a-x)+\frac{7}{3}\log(a+x)$. **6.** $2\log 15$.

7. $\log 3+\frac{3}{2}\log 2$. **8.** $\frac{2}{3}\log 3+\frac{2}{3}\log 7-2\log 2$.

9. $\log 5+2\log 7+6\log 2+4\log 3-3\log 11-\log 13$.

10. $\frac{1}{9}\log 2+\frac{4}{9}\log 3+\frac{1}{9}\log 23+\frac{2}{15}\log 100$.

11. 2·10721. **12.** 2·4014005. **13.** 2·4683473.
14. ·5563025. **15.** 2·0948204. **16.** 1·1672269.
17. ·1620254. **18.** ·4594421. **19.** ·1892631.
20. ·4867866. **21.** ·4146519. **22.** ·2385606.
23. ·150515. **24.** ·60206.

XX. p. 142.

1. 2; −1; −3; 0; −4; 1; 1; 1; 4.

2. $\bar{3}\cdot27955$; $\bar{1}\cdot766944$; $\bar{1}\cdot220854$. **3.** $\bar{2}\cdot782138$.

4. 5·1945698; 3·1945698, $\bar{3}\cdot1945698$, ·8054302, $\bar{3}\cdot805432$.

5. 4·6082371, $\bar{1}\cdot6082371$, 1·3041185, $\bar{2}\cdot3917629$, $\bar{1}\cdot7972543$.

6. $\bar{1}\cdot3222193$. **7.** $\bar{2}\cdot8864907$.
8. $\bar{1}\cdot3421227$. **9.** $\bar{2}\cdot1760913$.
10. $\bar{3}\cdot3636120$. **11.** $\bar{1}\cdot9793037$.
12. $\bar{1}\cdot8356949$. **13.** ·059837.
14. ·175124. **15.** $\bar{1}\cdot9915468$.
16. $\bar{1}\cdot849485$. **17.** $\bar{1}\cdot7614394$.
18. $\bar{1}\cdot9375306$. **19.** $\bar{1}\cdot8182503$. **20.** $\bar{1}\cdot7907879$.
21. 1·77. **22.** 2·57. **23.** −·37. **24.** 1·24.

XXI (*a*) p. 146.

1. 4·9845874. **2.** 4·6087535.
3. 2·3761852. **4.** $\bar{2}\cdot5012041$.
5. ·8028881. **6.** $\bar{3}\cdot9007201$.
7. 3·933398. **8.** 1·1943994.
9. 367·3442. **10.** ·8681168.
11. 4·671028. **12.** ·001819019.
13. 1·392461. **14.** 5301·362.
15. 60; ·5551895. **16.** 2998.
17. £1480 4*s.* 10*d.* **18.** £7,958 11*s.* 2*d.*
19. 24 years. **20.** 6·240323.

XXI (b) p. 150.

1. 9·7711652. **2.** 9·5827176. **3.** 9·8957889.
4. 10·1700067. **5.** 9·1773808. **6.** $61° 21' 38''$.
7. $30° 16' 30''$. **8.** $72° 40' 29''$. **9.** ·5687014.
10. 3·2240442. **11.** 1·638636. **12.** 2·485285.

XXII (b) p. 160.

7. $60°$. **8.** $45°$. **9.** $\sqrt{\frac{5}{3}}$. **10.** $\sqrt{\frac{8}{15}}$.

11. 1. **12.** $\frac{12}{37}$. **13.** $A = 60°, B = 75°, C = 45°$.

14. $135°$. **17.** $\frac{20}{21}$.

XXIII (a) p. 164.

1. $A = 30°, B = 60°, a = 8\sqrt{3}$. **2.** $\frac{9\sqrt{2}}{2}(\sqrt{3} - 1), \frac{9\sqrt{2}}{2}(\sqrt{3} + 1)$.

3. $51° 57' 19''$; $38° 2' 41''$. **4.** $a = 11·4621$; $c = 17·1299$.

5. $A = 57° 24' 44''$; $B = 32° 35' 16''$. **6.** $b = 200·8973$.

7. 2·3708911, 2·8979242. **8.** 3·8273784, 3·7353360.

XXIII (b) p. 166.

1. $C = 105°, b = 12$. **2.** $c = \sqrt{6} + \sqrt{2}, a = 2\sqrt{2}, A = 45°$.

3. $b = 17·32, C = 36° 25'$. **4.** $a = 10, B = 85° 50'$.

5. $b = 172·6436$. **6.** $c = 321·08$.

7. $A = 106° 15', b = 767·721, c = 1263·581$.

8. $b = 12414, c = 9017·97, C = 36° 17' 11''$.

9. The remaining angle is $180 + \beta - \gamma$, the other sides are $2a \sin \alpha \sin(\gamma - \alpha)$, $2a \sin \alpha \sin(\alpha - \beta)$.

XXIII (c) p. 171.

1. $c = 7$. **2.** $A = 105°, B = 15°, c = \sqrt{6}$.

3. Remaining side 5, angles $53° 8'$, $73° 44'$.

4. $B = 80° 27' 24''$, $C = 29° 32' 36''$. **5.** $C = 74° 50' 38''$, $A = 50° 32' 58''$.

6. $B = 148° 7' 50''$, $C = 6° 22' 10''$. **7.** $B = 123° 42' 50''$, $C = 12° 17' 10''$.

8. $B = 71° 44' 30''$, $C = 48° 15' 30''$, $a = 12·767$.

9. $B = 108° 36' 30''$, $C = 31° 23' 30''$.

10. $B = 109° 11' 14''$, $A = 33° 48' 46''$, $a = 307·0655$.

XXIII (*d*) p. 174.

1. $60°$. 2. $45°$. 3. $120°$.
4. $25° 12' 32''$. 5. $A = 16° 16'$; $B = 59° 32'$; $C = 104° 12'$.
6. $132° 34' 32''$. 7. $91° 4' 52''$. 8. $55° 46' 16''$.
9. $A = 19° 11' 17''$; $B = 61° 13' 4''$; $C = 99° 35' 39''$. 10. $63° 30' 58''$.

XXIII (*e*) p. 177.

1. $B = 60°$ or $120°$; $C = 75°$ or $15°$, $c = \sqrt{3} + 1$ or $\sqrt{3} - 1$.
2. $C = 45°$ or $135°$, $A = 105°$ or $15°$, $a = \frac{3}{2}(\sqrt{6} + \sqrt{2}), \frac{3}{2}(\sqrt{6} - \sqrt{2})$.
3. $B = 32° 21' 54''$, $C = 106° 28' 6''$. 4. $B = 49° 16' 5''$, $C = 10° 43' 55''$.
5. $B = 96° 27'$ or $19° 3'$. 6. $A = 21° 23'$, $B = 126° 22'$.
7. $B = 65° 59'$, $C = 41° 56' 12''$.

XXIV (*a*) p. 181.

1. 64·95 ft.; 177·45 ft. 2. 512·13 yds. 4. 25·7834 yds.
5. 6·34 miles. 7. 104·93 ft. 11. 63·578 yds.
12. $2\frac{1}{2}$ mi; $23° 8'$. 13. $78\frac{3}{4}$ ft., 175 ft. 14. 4·627 mi.
15. 2530 ft. 16. $\frac{25}{2}\sqrt{7}$ ft., $17\frac{1}{2}$ ft. 17. 229·149 yds.
18. $\tan^{-1}\dfrac{bc}{\sqrt{a^4 - b^4} - ac}$. 19. 19·0296 mi., 19·953 mi.
23. 221·5958. 24. 6·818 mi. per hour.
26. $\cot^{-1}\dfrac{(b \cot \alpha - a \cot \beta)}{a + b}$. 27. 1 mi., 1·21971 mi.
28. $\dfrac{c \cos(\alpha + \beta)}{\sin(\beta - \alpha)}$. 30. $\dfrac{a\sqrt{3} + b\sqrt{2}}{2}$, $45°$.
31. 1514·397 yds., 4163·65 yds.

XXIV (*b*) p. 187.

3. 185·1516. 6. 83·414 yds.
10. 670·8 yds. 12. 2036·33 yds.

XXIV (*c*) p. 190.

1. 40 mi. 2. 4224 mi. 3. 7·775252 mi.
5. 29·71 yds. 6. 36·7226 mi.

XXV (*a*) p. 192.

1. 6 sq. ft. 2. $17\frac{1}{2}$ sq. ft. 3. $6 + 2\sqrt{3}$ sq. in
4. ·54 sq. in. 5. $5000\sqrt{3}$, $2500\sqrt{3}$.
6. 28·47717 sq. in. 11. 4, 6, 8 in.

XXV (*b*) p. 194.

1. $2\frac{1}{4}$ in. 2. $2\cdot309$ ft.
3. $20\cdot8\dot{3}$ in. 4. $56\cdot5794$ in.

XXV (*c*) p. 196.

1. 9.

XXVI. p. 205.

1. $4\sqrt{3}$. 2. 7; $\sqrt{5}$. 3. 6 ft., 120°.
4. $16\sqrt{2} - 16$, 135°. 5. $\cdot106$ sq. in. 9. πa^2.
10. $\cdot85$ in.

XXVII. p. 210.

16. 3. 17. $\frac{1}{2}$. 18. $0, \frac{1}{2}$.
19. $(2n+1)\frac{\pi}{2}, \left(2n \pm \frac{2}{3}\right)\pi$. 20. $\pm ab$. 23. 0.

Miscellaneous Examples, p. 212.

1. $20\frac{5}{11}$°. 2. 1.
3. $B = 60°$, $C = 90°$, $c = 60$, or $B = 120°$, $C = 30°$, $c = 30$.
6. (1) $(2n+1)\frac{\pi}{2}, n\pi \pm \frac{\pi}{3}$. (2) $(2n+1)\frac{\pi}{10}, (2n+1)\frac{\pi}{2}$.
8. 55° 46′ 16″. 12. $26\frac{4}{121}$ min. past twelve.
15. (1) $2n\pi, 2n\pi \pm \frac{2\pi}{3}$. (2) $n\pi, n\pi \pm \frac{\pi}{6}$. 22. $5\frac{37}{83}$ ft.
25. $0, \frac{1}{2}$. 26. $600(\sqrt{3} - 1)$ yds., $300(\sqrt{6} - \sqrt{2})$ yds
31. (1) $(2n+1)\pi, (2n+1)\frac{\pi}{2}$. (2) $n\pi \pm \frac{\pi}{6}$.
38. $\frac{25}{7}$. 42. 82° 37′; 49° 31′.
44. $\cdot501717$ 47. $\pm \operatorname{cosec} \phi$.
49. $\frac{-\sqrt{3} - 1}{2\sqrt{2}}$. 51. 33° 18′ 26″.
53. $\frac{2n+1}{8}\pi, \frac{2}{3}n\pi \pm \frac{\pi}{9}$. 58. $4\sqrt{2}$.
59. $121\cdot24$ ft., $76\ 88\cdot$ft. 62. $1272\frac{4}{7}$ cub. in.

65. 75. **67.** $\sqrt{2}$, 2. **70.** $\sin A = \frac{a}{b}$.

72. $\cos\phi = \frac{1 \pm \sqrt{5}}{8}$; $\cos^2\theta = \frac{5 \pm \sqrt{5}}{8}$.

78. $\frac{a}{4 \sin A}$. **80.** $\cdot978$, $\cdot669$, $-\frac{1}{\sqrt{3}}$, -2.

82. $(\sin^2 x - 2\sin x + 2)(\sin^2 x + 2\sin x + 2)$.

86. $a\sqrt{c^2 - b^2} + b\sqrt{c^2 - a^2} = c^2$. **87.** $n\pi \pm \alpha$.

90. $x^2 + y^2 = a^2$ **97.** $y^2 - x^2 = 8a^2$. **102.** $b^2 = a^2 + c$.

EXERCISES, p. 221.

1.

1. 7°. **4.** 60 ft. **5.** 60°.

6. $\frac{a^2 + b^2}{a^2 - b^2}$. **7.** $\sin A = \frac{m-n}{2}$, $\cos A = \frac{m-n}{m+n}$.

10. A = 45°, B = 60°.

2.

1. 14, 15. **4.** 96°. **6.** $m - n = 0$. **7.** 45°.

8. $\frac{a^2 - b^2}{a^2 + b^2}$, $\frac{2ab}{a^2 - b^2}$. **9.** 150 sq. in. **10.** $86\frac{2}{5}$ yds., $115\frac{1}{5}$ yds., $64\frac{4}{5}$ yds.

3.

1. 40°, 60°, 80°. **2.** 45°.

3. (1) $n\pi, 2n\pi \pm \frac{\pi}{3}$. (2) $(2n+1)\pi, \frac{2}{3}n\pi + \frac{\pi}{6}$.

8. $\sqrt{3}$, $-\frac{1}{\sqrt{3}}$, -2, ∞. **9.** 138·58 ft.

4.

1. 170°, $\frac{17}{18}\pi$. **2.** (1) $n\pi$, $n\pi + \tan^{-1} 2$. (2) $(2n+1)\frac{\pi}{2}$, $n\pi + \frac{\pi}{4}$.

3. $120(\sqrt{3} + 1)$. **10.** 5·656 mi.

5.

1. 120° 40′ $54\frac{6}{11}$″. **2.** (1) $n\pi \pm \frac{\pi}{4}$. (2) $\frac{n\pi}{2} + (-1)^n \frac{\pi}{12}$.

4. 86·6 ft., 50 ft. **5.** 41° 17′ 8·5″; 48° 42′ 51·5″.

9. $-\frac{1}{2}, 1, -\sqrt{2}, -2$. **10.** $a^{\frac{1}{3}} b^{\frac{1}{3}} (a^{\frac{2}{3}} + b^{\frac{2}{3}})^{\frac{1}{2}}$; $(a^{\frac{2}{3}} + b^{\frac{2}{3}})^{\frac{3}{2}}$.

6.

2. 262 ft. 6 in.

5. $70° \ 53' \ 40''$, $49° \ 6' \ 20''$.

6. (1) $n\pi \pm \frac{\pi}{3}$. (2) $n\pi, 2n\pi \pm \frac{\pi}{3}$.

8. 1·06.

7.

1. 9:2.

3. 508·938.

4. $(b-c)^2 \operatorname{cosec}^2 2\alpha + (b+c-\cos\phi)^2 \sec^2 2\alpha = 1$.

5. $5(3-\sqrt{3}) = 6 \cdot 34$ mi.

6. (1) $n\pi + \frac{\pi}{4}, n\pi + \frac{\pi}{8}, n\pi + \frac{5\pi}{8}$. (2) $n\pi + \frac{3\pi}{4}$. (3) $n\pi, n\pi + (-1)^n \frac{\pi}{3}$.

9. $\frac{3}{7}\sqrt{5}, \frac{1}{3}\sqrt{5}, \frac{4}{21}\sqrt{5}$.

8.

1. $54° \ 49' \ 49 \cdot 5''$.

2. (1) $-\sqrt{1+\sin A} + \sqrt{1-\sin A}$. (2) $\sqrt{1+\sin A} - \sqrt{1-\sin A}$.

4. (1) $n\pi \pm \frac{\pi}{3}$. (2) $\frac{n\pi}{3}, \frac{n\pi}{5}$. (3) $(2n+1)\pi$; $n\pi + \frac{\pi}{4}$.

9. $22° \ 37' \ 11''$; $67° \ 22' \ 49''$; $90°$.

9.

4. 45°, 60°, 75°.

5. $2n\pi \pm \frac{\pi}{6}, 2n\pi \pm \frac{2\pi}{3}$.

7. (1) $n\pi \pm \frac{\pi}{12}$. (2) $n\pi + (-1)^n \frac{\pi}{4}$.

8. $68° \ 25' \ 18''$; $37° \ 14' \ 42''$.

9. 6.

10.

1. 9, 10.

2. 55 ft. 5in.

8. $\frac{a \sin\beta \sin 2\alpha}{\sin(2\alpha - \beta)}$.

9. $6, 12, 4\sqrt{3}, 6(\sqrt{6}-\sqrt{2})$.

10. Ratios of sides $1 : 2 : \sqrt{3}$.

11.

2. (1) $n\pi, n\pi \pm \frac{\pi}{4}$. (2) $(2n+1)\frac{\pi}{2}, \frac{n\pi}{2} \pm \frac{\pi}{24}$. (3) $n\pi + \frac{\pi}{4}, n\pi + \frac{\pi}{12}, n\pi + \frac{5\pi}{12}$.

6. $36° \ 7' \ 47''$.

12.

1. 535 mi.

3. (1) $(2n+1)\pi, 2n\pi \pm \frac{\pi}{3}$. (2) $n\pi, n\pi \pm \frac{\pi}{6}$. (3) $(2n+1)\frac{\pi}{2}, \frac{2m\pi}{n-1} \pm \frac{\pi}{3(n-1)}$.

13.

1. $\frac{3}{4}$. 4. (1) $n\pi \pm \alpha$. (2) $n\pi + (-1)^n \frac{3\pi}{20}, n\pi + (-1)^n \frac{11\pi}{20}$.

6. 4·8 mi. 8. $4 \sin 2x \cos 3x \cos 4x$.

14.

3. 78° 27′ 47″. 7. $0, \frac{\sqrt{3}-1}{2\sqrt{2}}, -2+\sqrt{3}$. 8. $\frac{12}{5}, \frac{4}{3}$.

9. (1) $(2n+1)\frac{\pi}{2}, x = \frac{1}{2}\cos^{-1}\left(\frac{\pm\sqrt{5}-1}{2}\right)$. (2) $0, \frac{1}{4}$. (3.) $n\pi, n\pi \pm \frac{\pi}{6}$.

15.

1. 9 or 16. 3. (1) $\frac{n\pi}{2}, \frac{n\pi}{2} \pm \frac{\pi}{12}$. (2) $\frac{2}{3}n\pi \pm \frac{\pi}{15}$. (3) $n\pi \pm \frac{\pi}{2(a+b)}$.

5. 313·4605. 8. 49·09 in.

16.

3. 41° 16′ 51·5″

6. (1) $x = n\pi + \frac{\pi}{12}$ or $n\pi + \frac{5\pi}{12}$; $y = n\pi + \frac{5\pi}{12}$ or $n\pi + \frac{\pi}{12}$. (2) $n\pi, n\pi + \frac{3\pi}{4}$

17.

2. (1) $\frac{n}{2}\pi + \frac{\pi}{6}$ (2) $n\pi, 2n\pi + \alpha$. 5. $a^{\frac{2}{3}} b^{\frac{2}{3}} \left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right) = 1$.

6. $\tan\left(\frac{\pi}{4} - \frac{\alpha}{2}\right) \tan\left(\frac{\pi}{4} - \frac{\beta}{2}\right)$.

18.

3. (1) $\sin\frac{1}{2}(\sin^{-1}\alpha + \sin^{-1}\beta)$. (2) $0, \sqrt{2}$.

4. 140 yds. 8. (1.) $b = c$. (2.) $b + c = \pm a\sqrt{2}$.

19.

1. $a^4 + b^4 = 2a^2$.

3. (1) $\frac{n\pi}{2}, n\pi, \frac{n\pi}{5}$. (2) $\frac{n\pi}{4}, \frac{2n\pi}{3} \pm \frac{\pi}{9}$. (3) $n\pi, 2n\pi \pm \frac{\pi}{4}, 2n\pi \pm \cos^{-1}\left(-\frac{1}{2\sqrt{2}}\right)$

5. $A = 105°, B = 15°, c = \sqrt{6}$. 10. 25·2982.

20.

2. (1) $\begin{cases} x = n\pi;\ n\pi + (-1)^n \sin^{-1}\frac{3}{5\sqrt{2}} \\ y = m\pi;\ m\pi + \tan^{-1}\frac{1}{7} \end{cases}$. (2) $x = \frac{\sqrt{17}+3}{4}, y = \frac{\sqrt{17}-3}{4}$.

3. $x = a\cos(\beta - \alpha)$.

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